

# DIFFERENTIAL FORMS, FUKAYA $A_\infty$ ALGEBRAS, AND GROMOV-WITTEN AXIOMS

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ABSTRACT. Consider the differential forms  $A^*(L)$  on a Lagrangian submanifold  $L \subset X$ . Following ideas of Fukaya-Oh-Ohta-Ono, we construct a family of cyclic unital curved  $A_\infty$  structures on  $A^*(L)$ , parameterized by the cohomology of  $X$  relative to  $L$ . The family of  $A_\infty$  structures satisfies properties analogous to the axioms of Gromov-Witten theory. Our construction is canonical up to  $A_\infty$  pseudo-isotopy. We assume moduli spaces and boundary evaluation maps are regular and thus we do not use the theory of the virtual fundamental class.

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## 1. INTRODUCTION

In the beautiful series of papers [1, 2, 4–6], Fukaya and Fukaya-Oh-Ohta-Ono reworked and extended in the language of differential forms the theory of  $A_\infty$  algebras associated to Lagrangian submanifolds from their book [3]. With the help of this new tool, they obtained many striking results in Floer theory and mirror symmetry. They work in a very general setting, and introduce fundamental new ideas in the theory of the virtual fundamental class to address the technical difficulties that arise.

The present paper uses differential forms to construct a family of cyclic unital curved  $A_\infty$  algebras associated to a Lagrangian submanifold. We assume the moduli spaces and boundary evaluation maps are regular, so virtual fundamental class techniques are not necessary.

Our family of  $A_\infty$  algebras is parameterized by the cohomology of  $X$  relative to  $L$ , as opposed to absolute cohomology of  $X$  as found in the literature. The family satisfies differential equations analogous to the fundamental class and divisor axioms of Gromov-Witten theory. Our definition of unitality is stronger than the standard one. The use of relative cohomology is of crucial importance for proving unitality and the divisor equation.

We use the framework developed here in [16, 17] to define open Gromov-Witten invariants and establish their properties. For this purpose, we also include a discussion of the operator  $\mathbf{m}_{-1}$  as defined in [2].

**1.1. Setting.** Consider a symplectic manifold  $(X, \omega)$  of  $\dim_{\mathbb{R}} X = 2n$ , and a connected, relatively-spin Lagrangian submanifold  $L$ . Let  $J$  be an  $\omega$ -tame almost complex structure on  $X$ . Denote by  $\mu : H_2(X, L) \rightarrow \mathbb{Z}$  the Maslov index. Denote by  $A^*(L)$  the algebra of differential forms on  $L$  with coefficients in  $\mathbb{R}$ . Let  $\Pi$  be a quotient of  $H_2(X, L; \mathbb{Z})$  by a possibly trivial subgroup contained in the kernel of the homomorphism  $\omega \oplus \mu : H_2(X, L; \mathbb{Z}) \rightarrow \mathbb{R} \oplus \mathbb{Z}$ . Thus the homomorphisms  $\omega, \mu$ , descend to  $\Pi$ . Denote by  $\beta_0$  the zero element of  $\Pi$ . We use a Novikov ring  $\Lambda$  which is a completion of a subring of the group ring of  $\Pi$ . The precise definition follows. Denote by  $T^\beta$  the element of the group ring corresponding to  $\beta \in \Pi$ , so  $T^{\beta_1} T^{\beta_2} = T^{\beta_1 + \beta_2}$ . Then,

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\beta_i} \mid a_i \in \mathbb{R}, \beta_i \in \Pi, \omega(\beta_i) \geq 0, \lim_{i \rightarrow \infty} \omega(\beta_i) = \infty \right\}.$$

A grading is defined on  $\Lambda$  by declaring  $T^\beta$  to be of degree  $\mu(\beta)$ .

For  $k \geq -1$ , denote by  $\mathcal{M}_{k+1,l}(\beta)$  the moduli space of genus zero  $J$ -holomorphic open stable maps to  $(X, L)$  of degree  $\beta \in \Pi$  with one boundary component,  $k+1$  boundary marked points and  $l$  interior marked points. The boundary points are labeled according to their cyclic order. Denote by  $evb_i : \mathcal{M}_{k+1,l}(\beta) \rightarrow L$ , and  $evi_j : \mathcal{M}_{k+1,l}(\beta) \rightarrow X$ , the boundary and interior evaluation maps respectively, where  $i = 0, \dots, k$ , and  $j = 1, \dots, l$ . Assume that  $\mathcal{M}_{k+1,l}(\beta)$  is a smooth orbifold with corners. Then it carries a natural orientation induced by the relative spin structure on  $(X, L)$ , as in [3, Chapter 8]. Assume in addition that  $evb_0$  is a proper submersion. See Example 1.4 for a discussion and examples of when these assumptions hold.

Let  $t_0, \dots, t_N$ , be formal variables with even degrees. For  $m > 0$  denote by  $A^m(X, L)$  the differential  $m$ -forms on  $X$  that vanish on  $L$ , and denote by  $A^0(X, L)$  the functions on  $X$  that are constant on  $L$ . The exterior differential  $d$  makes  $A^*(X, L)$  into a complex. Set

$$\begin{aligned} R &:= \Lambda[[t_0, \dots, t_N]], \quad Q := \mathbb{R}[t_0, \dots, t_N], \\ C &:= A^*(L) \otimes R, \quad \text{and} \quad D := A^*(X, L) \otimes Q, \end{aligned}$$

where  $\otimes$  is understood as the completed tensor product. Write  $\hat{H}^*(X, L; Q) = H^*(D)$ . The gradings on  $C, D$ , and  $\hat{H}^*(X, L; Q)$ , take into account the degrees of  $t_j, T^\beta$ , and the degree of differential forms.

Define a valuation

$$\nu : R \longrightarrow \mathbb{R},$$

by

$$\nu \left( \sum_{j=0}^{\infty} a_j T^{\beta_j} s^{k_j} \prod_{a=1}^N t_a^{l_{aj}} \right) = \inf_{\substack{j \\ a_j \neq 0}} \left( \omega(\beta_j) + k_j + \sum_{a=1}^N l_{aj} \right).$$

The valuation  $\nu$  induces a valuation on  $C$  and its tensor products, which we also denote by  $\nu$ . Define  $\mathcal{I}_R := \{\alpha \in R \mid \nu(\alpha) > 0\}$ . Then  $\bar{R} := R/\mathcal{I}_R = \mathbb{R}$  and

$$\bar{C} := C/(\mathcal{I}_R C) = A^*(L).$$

**1.2. Statement of results.** Let  $\mathcal{R}$  be a differential graded algebra with valuation  $\varsigma_{\mathcal{R}}$  and let  $\mathcal{C}$  be a graded module over  $\mathcal{R}$  with valuation  $\varsigma_{\mathcal{C}}$ .

**Definition 1.1.** An  $n$ -dimensional (curved) **cyclic unital  $A_\infty$  structure** on  $\mathcal{C}$  is a triple  $(\{m_k\}_{k \geq 0}, \prec, \succ, \mathbf{e})$ , where  $\mathbf{m}_k : \mathcal{C}^{\otimes k} \rightarrow \mathcal{C}[2-k]$  are  $\mathcal{R}$ -linear operations,  $\prec, \succ : \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{R}[-n]$  is  $\mathcal{R}$ -linear, and  $\mathbf{e} \in \mathcal{C}$  with  $\deg_{\mathcal{C}} \mathbf{e} = 0$ , satisfying the following properties.

(1) The  $A_\infty$  relations hold:

$$\sum_{\substack{k_1+k_2=k+1 \\ 1 \leq i \leq k_1}} (-1)^{\sum_{j=1}^{i-1} (\deg_{\mathcal{C}} \alpha_j + 1)} \mathbf{m}_{k_1}(\alpha_1, \dots, \alpha_{i-1}, \mathbf{m}_{k_2}(\alpha_i, \dots, \alpha_{i+k_2-1}), \alpha_{i+k_2}, \dots, \alpha_k) = 0.$$

(2)  $\varsigma_{\mathcal{C}}(\mathbf{m}_k(\alpha)) \geq \varsigma_{\mathcal{C}}(\alpha)$  and  $\varsigma_{\mathcal{C}}(\mathbf{m}_0) > 0$ .

(3)  $\varsigma_{\mathcal{R}}(\prec \alpha_1, \alpha_2 \succ) \geq \varsigma_{\mathcal{C}}(\alpha_1) + \varsigma_{\mathcal{C}}(\alpha_2)$ .

(4)  $\prec \alpha_1, \alpha_2 \succ = (-1)^{(\deg_{\mathcal{C}} \alpha_1 + 1)(\deg_{\mathcal{C}} \alpha_2 + 1) + 1} \prec \alpha_2, \alpha_1 \succ$ .

(5) The pairing is cyclic:

$$\begin{aligned} \prec \mathbf{m}_k(\alpha_1, \dots, \alpha_k), \alpha_{k+1} \succ &= \\ &= (-1)^{(\deg_{\mathcal{C}} \alpha_{k+1} + 1) \sum_{j=1}^k (\deg_{\mathcal{C}} \alpha_j + 1)} \prec \mathbf{m}_k(\alpha_{k+1}, \alpha_1, \dots, \alpha_{k-1}), \alpha_k \succ + \delta_{1,k} \cdot d \prec \alpha_1, \alpha_2 \succ. \end{aligned}$$

(6)  $\prec \mathbf{m}_k(\alpha_1, \dots, \alpha_k), \mathbf{e} \succ = 0 \quad \forall k \neq 1, 2$ .

(7)  $\prec \mathbf{m}_1(\alpha), \mathbf{e} \succ = d \prec \alpha, \mathbf{e} \succ$ .

(8)  $\mathbf{m}_2(\mathbf{e}, \alpha) = \alpha = (-1)^{\deg_{\mathcal{C}} \alpha} \mathbf{m}_2(\alpha, \mathbf{e})$ .

*Remark 1.2.* Our definition differs from that of [1, 4, 5] in that  $\mathbf{m}_0$  is required to respect the unit  $\mathbf{e}$ .

Equip  $R$  with the trivial differential  $d_R = 0$ . Consider the  $R$ -module  $C$ . For  $\gamma \in D$  with  $d\gamma = 0$  and  $\deg_D \gamma = 2$ , define maps

$$\mathbf{m}_k^{\gamma, \beta}, \mathbf{m}_k^\gamma : C^{\otimes k} \longrightarrow C$$

for  $k \geq 0$  by

$$\begin{aligned} \mathbf{m}_k^{\gamma, \beta}(\alpha_1, \dots, \alpha_k) &:= \delta_{k,1} \cdot d\alpha_1 + (-1)^{\sum_{j=1}^k j(|\alpha_j|+1) + nk+1} \sum_{l \geq 0} \frac{1}{l!} \text{ev} b_{0*}^\beta \left( \bigwedge_{j=1}^k (\text{ev} b_j^\beta)^* \alpha_j \wedge \bigwedge_{j=1}^l (\text{ev} i_j^\beta)^* \gamma \right), \\ \mathbf{m}_k^\gamma &:= \sum_{\beta \in \Pi} T^\beta \mathbf{m}_k^{\gamma, \beta}. \end{aligned}$$

Denote by  $\langle \cdot, \cdot \rangle$  the signed Poincaré pairing

$$\langle \xi, \eta \rangle := (-1)^{|\eta|} \int_L \xi \wedge \eta. \quad (1)$$

Denote by 1 the constant function  $1 \in A^0(L)$ .

**Theorem 1.** *The triple  $(\{\mathbf{m}_k^\gamma\}_{k \geq 0}, \langle \cdot, \cdot \rangle, 1)$  is a cyclic unital  $A_\infty$  structure on  $C$ .*

Set  $\mathfrak{R} := A^*([0, 1]; R)$ ,  $\mathfrak{C} := A^*([0, 1] \times L; R)$ , and  $\mathfrak{D} := A^*([0, 1] \times X, [0, 1] \times L; Q)$ . The valuation  $\nu$  induces valuations on  $\mathfrak{R}, \mathfrak{C}$ , and  $\mathfrak{D}$ , which we still denote by  $\nu$ . For  $t \in [0, 1]$ , denote by

$$j_t : L \rightarrow [0, 1] \times L$$

the inclusion  $j_t(p) = (t, p)$ .

**Definition 1.3.** Let  $S_1 = (\mathbf{m}, \prec, \succ, \mathbf{e})$  and  $S_2 = (\mathbf{m}', \prec, \succ, \mathbf{e}')$  be cyclic unital  $A_\infty$  structures on  $C$ . A cyclic unital **pseudo-isotopy** from  $S_1$  to  $S_2$  is a cyclic unital  $A_\infty$  structure  $(\tilde{\mathbf{m}}, \preccurlyeq, \succcurlyeq, \tilde{\mathbf{e}})$  on the  $\mathfrak{R}$ -module  $\mathfrak{C}$  such that for all  $\tilde{\alpha}_j \in \mathfrak{C}$  and all  $k \geq 0$ ,

$$\begin{aligned} j_0^* \tilde{\mathbf{m}}_k(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k) &= \mathbf{m}_k(j_0^* \tilde{\alpha}_1, \dots, j_0^* \tilde{\alpha}_k), \\ j_1^* \tilde{\mathbf{m}}_k(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k) &= \mathbf{m}'_k(j_1^* \tilde{\alpha}_1, \dots, j_1^* \tilde{\alpha}_k), \end{aligned}$$

and

$$\begin{aligned} j_0^* \preccurlyeq \tilde{\alpha}_1, \tilde{\alpha}_2 \succcurlyeq &= \prec j_0^* \tilde{\alpha}_1, j_0^* \tilde{\alpha}_2 \succ, & j_0^* \tilde{\mathbf{e}} &= \mathbf{e}, \\ j_1^* \preccurlyeq \tilde{\alpha}_1, \tilde{\alpha}_2 \succcurlyeq &= \prec j_1^* \tilde{\alpha}_1, j_1^* \tilde{\alpha}_2 \succ', & j_1^* \tilde{\mathbf{e}} &= \mathbf{e}'. \end{aligned}$$

**Theorem 2.** *Let  $\gamma, \gamma' \in D$  be closed with  $\deg_D \gamma = \deg_D \gamma' = 2$ . If  $[\gamma] = [\gamma'] \in \hat{H}^*(X, L; Q)$ , then there exists a cyclic unital pseudo-isotopy from  $(\mathbf{m}^\gamma, \langle \cdot, \cdot \rangle, 1)$  to  $(\mathbf{m}^{\gamma'}, \langle \cdot, \cdot \rangle, 1)$ .*

In Section 4 we also discuss pseudo-isotopies arising from varying  $J$ . By property (2), the maps  $\mathbf{m}_k$  descend to maps on the quotient

$$\bar{\mathbf{m}}_k : \bar{C}^{\otimes k} \longrightarrow \bar{C}.$$

**Theorem 3.** *Suppose  $\partial_{t_0} \gamma = 1 \in A^0(X, L) \otimes Q$  and  $\partial_{t_1} \gamma = \gamma_1 \in A^2(X, L) \otimes Q$ . Then the operations  $\mathbf{m}_k^\gamma$  satisfy the following properties.*

- (1) (Fundamental class)  $\partial_{t_0} \mathbf{m}_k^\gamma = -1 \cdot \delta_{0,k}$ .
- (2) (Divisor)  $\partial_{t_1} \mathbf{m}_k^{\gamma, \beta} = \int_\beta \gamma_1 \cdot \mathbf{m}_k^{\gamma, \beta}$ .

- (3) (*Energy zero*) The operations  $\mathbf{m}_k^\gamma$  are deformations of the usual dg-algebra structure on differential forms. That is,

$$\bar{\mathbf{m}}_1^\gamma(\alpha) = d\alpha, \quad \bar{\mathbf{m}}_2^\gamma(\alpha_1, \alpha_2) = (-1)^{|\alpha_1|} \alpha_1 \wedge \alpha_2, \quad \bar{\mathbf{m}}_k^\gamma = 0, \quad k \neq 1, 2.$$

In Section 2.2, we also construct a distinguished element  $\mathbf{m}_{-1}^\gamma \in R$  following [2]. In the subsequent sections, we prove its properties along with the properties of  $\mathbf{m}_k^\gamma$  for  $k \geq 0$ . In Section 4 we construct  $\tilde{\mathbf{m}}_{-1}^\gamma$ , the analogous structure for a pseudo-isotopy. In Section 4.3 we reformulate the  $A_\infty$  structure equations of the pseudo-isotopy so that the structure equation for  $\tilde{\mathbf{m}}_{-1}^\gamma$  fits more naturally. The reformulated  $A_\infty$  structure equations are used in [17] to prove the superpotential is invariant under pseudo-isotopy.

*Example 1.4.* Suppose  $J$  is integrable and suppose there exists a Lie group  $G_X$  that acts transitively on  $X$  by  $J$ -holomorphic diffeomorphisms. Furthermore, suppose there exists a subgroup  $G_L \subset G_X$  that preserves  $L$  and acts transitively on  $L$ . Then our assumptions that  $\mathcal{M}_{k,l}(\beta)$  is a smooth orbifold with corners and  $evb_0$  is a submersion are satisfied.

Indeed, [12, Proposition 7.4.3] shows that all  $J$ -holomorphic genus zero stable maps to  $X$  without boundary are regular. A small modification of the argument there shows that  $J$ -holomorphic genus zero stable maps to  $(X, L)$  with one boundary component are also regular. For regularity of holomorphic disks, instead of Grothendieck's classification [7], one uses Oh's work on the Riemann-Hilbert problem [13]. The argument applies equally well to maps that are not somewhere injective in the sense of [12, Section 2.5]. So, the fact that a  $J$ -holomorphic map from a domain with boundary need not factor through a somewhere injective map [10, 11] does not affect the argument. Once all stable maps are regular, one modifies the techniques of [15] to show that the moduli space is a smooth orbifold with corners. Since  $G_L$  acts transitively on  $L$ , it follows that  $evb_0$  is a submersion.

Thus, examples of  $(X, L)$  which satisfy our assumptions include  $(\mathbb{C}P^n, \mathbb{R}P^n)$  with the standard complex and symplectic structures or, more generally, flag varieties, Grassmannians, and products thereof. Virtual fundamental class techniques should allow the extension of our results to general target manifolds.

**1.3. Outline.** In Section 2.1 we review orientation conventions and properties of the push-forward of differential forms. Sections 2.2-2.4 formulate and prove the  $A_\infty$  structure relations for the closed-open maps  $\mathbf{q}_{k,l}$  for  $k \geq -1$ . In Section 3 we formulate and prove properties satisfied by the  $\mathbf{q}$  operators. The section closes with the proofs of Theorems 1 and 3. Section 4 constructs pseudo-isotopies and uses them to prove Theorem 2. Section 4.3 reformulates the  $A_\infty$  structure relations in a way that incorporates  $\mathbf{m}_{-1}$  more naturally.

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### 1.5. Notation.

We write  $I := [0, 1]$  for the closed unit interval.

Use  $i$  to denote the inclusion  $i : L \hookrightarrow X$ . By abuse of notation, we also use  $i$  for  $\text{Id} \times i : I \times L \rightarrow I \times X$ . The meaning in each case should be clear from the context.

Denote by  $pt$  the map (from any space) to a point.

Whenever a tensor product is written, we mean the completed tensor product. For example,  $A^*(L) \otimes R$  is the completion of the tensor product  $A^*(L) \otimes_{\mathbb{R}} R$  with respect to  $\nu$ .

Write  $A^*(L; R)$  for  $A^*(L) \otimes R$ . Similarly,  $A^*(X; Q)$  and  $A^*(X, L; Q)$  stand for  $A^*(X) \otimes Q$  and  $A^*(X, L) \otimes Q$  respectively.

Given  $\alpha$ , a homogeneous differential form with coefficients in  $R$ , denote by  $|\alpha|$  the degree of the differential form, ignoring the grading of  $R$ .

For non-homogeneous  $\alpha$ , denote by  $(\alpha)_j$  the form that is the part of degree  $j$  in  $\alpha$ . In particular,  $|(\alpha)_j| = j$ .

Denote by  $\bar{A}^k(\cdot)$  space of currents of cohomological degree  $k$ , i.e., the dual space of the differential forms  $A^{top-k}(\cdot)$ . Differential forms are identified as a subspace of currents by

$$\begin{aligned}\varphi : A^k(\cdot) &\hookrightarrow \bar{A}^k(\cdot), \\ \varphi(\eta)(\alpha) &= \int \eta \wedge \alpha, \quad \alpha \in A^{top-k}(\cdot).\end{aligned}$$

Accordingly, for a general current  $\zeta$ , we may use the notation  $\zeta(\alpha) = \int \zeta \wedge \alpha$ . For a current  $\zeta$ , denote by  $d\zeta$  the current that is characterized by  $d\zeta(\alpha) = (-1)^{1+|\zeta|}\zeta(d\alpha)$ . Thus  $d\varphi(\eta) = \varphi(d\eta)$ . Given  $f : M \rightarrow N$  and  $\zeta \in \bar{A}^*(M)$ , define the push-forward of  $\zeta$  by

$$(f_*\zeta)(\xi) = (-1)^{s \cdot m} \zeta(f^*\xi), \quad \xi \in A^m(N),$$

with  $s = \dim M - \dim N$ . So, when  $f$  is a submersion,  $f_*\varphi(\eta) = \varphi(f_*\eta)$ .

## 2. CONSTRUCTION

**2.1. Orientations and integration.** This section deals with orientation conventions and properties of integration for oriented orbifolds with corners. We follow the conventions of [8] concerning manifolds with corners.

Let  $M$  be an oriented orbifold with corners and let  $\iota : \partial M \rightarrow M$  denote the natural map. Our convention for the induced orientation on  $\partial M$  is this. Let  $p \in \partial M$  and let  $B$  be a basis for  $T_p \partial M$ . Let  $N \in T_{\iota(p)} M$  be the outward-pointing normal at  $p$ . We say  $B$  is positive if  $(N, B)$  is a positive basis for  $T_{\iota(p)} M$ .

To orient fiber products, and in particular fibers, we use the following convention. Let  $f : M \rightarrow N$  and  $g : P \rightarrow N$  be submersions, and let  $(p, q) \in M \times_N P$  with  $f(p) = g(q) = x$ . Let  $B_N$  be a positive basis of  $T_x N$ . Let  $B_M^2, B_P^1$ , be lists of linearly independent vectors in  $T_p M, T_q P$ , respectively, that are mapped to  $B_N$  by  $df_p, dg_q$ , respectively. Let  $B_M^1, B_P^2$ , be lists so that  $B_M = (B_M^1, B_M^2)$  and  $B_P = (B_P^1, B_P^2)$  are positive bases for  $T_p M$  and  $T_q P$  respectively. Identify

$$T_{(p,q)}(M \times_N P) \simeq \text{Ker}(df_p \oplus (-dg)_q : T_p M \oplus T_q P \longrightarrow T_x N).$$

Then the orientation on  $M \times_N P$  at  $(p, q)$  is given by

$$(B_M^1 \oplus (0, \dots, 0), B_M^2 \oplus B_P^1, (0, \dots, 0) \oplus B_P^2),$$

where the direct sum of two lists is taken componentwise. For a submersion  $h : Q \rightarrow S$ , and  $y \in S$ , we orient the fiber  $h^{-1}(y)$  by identifying it with the fiber product  $\{y\} \times_S Q$ . The preceding orientation conventions determine the signs in properties (2)-(4) below as well as Stokes' theorem, Proposition 2.2.

Let  $f : M \rightarrow N$  be a proper submersion of relative dimension  $s = \dim M - \dim N$ . Denote by  $f_* : A^*(M) \rightarrow A^*(N)[-s]$  the push-forward of forms along  $f$ , that is, integration over the fiber. We will need the following properties of  $f_*$  formulated in [9, Section 3.1].

### Proposition 2.1.

(1) Let  $f : M \rightarrow pt$  and  $\alpha \in A^m(M)$ . Then

$$f_*\alpha = \begin{cases} \int_M \alpha, & m = \dim M, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Let  $g : P \rightarrow M$ ,  $f : M \rightarrow N$ , be proper submersions. Then

$$f_* \circ g_* = (f \circ g)_*.$$

(3) Let  $f : M \rightarrow N$  be a proper submersion,  $\alpha \in A^*(N)$ ,  $\beta \in A^*(M)$ . Then

$$f_*(f^*\alpha \wedge \beta) = \alpha \wedge f_*\beta.$$

(4) Let

$$\begin{array}{ccc} P \times_N M & \xrightarrow{p} & M \\ \downarrow q & & \downarrow f \\ P & \xrightarrow{g} & N \end{array}$$

be a pull-back diagram of smooth maps, where  $f$  is a proper submersion. Let  $\alpha \in A^*(M)$ . Then

$$q_*p^*\alpha = g_*f_*\alpha.$$

Properties (1),(3), and (4), uniquely determine  $f_*$ . Furthermore, we have the following generalization of Stokes' theorem [9, Section 3.1].

**Proposition 2.2** (Stokes' theorem). *Let  $f : M \rightarrow N$  be a proper submersion of relative dimension  $s$ , and let  $\xi \in A^t(M)$ . Then*

$$d(f_*\xi) = f_*(d\xi) + (-1)^{s+t}(f|_{\partial M})_*\xi,$$

where  $\partial M$  is understood as the fiberwise boundary with respect to  $f$ .

**2.2. Formulation.** In this section, we construct a family of  $A_\infty$  structures following [1, 3, 6].

Denote by

$$\mathcal{M}_{k+1,l}(\beta) = \mathcal{M}_{k+1,l}(\beta; J)$$

the moduli space of genus zero  $J$ -holomorphic stable maps  $u : (\Sigma, \partial\Sigma) \rightarrow (X, L)$  with one boundary component, of degree

$$[u_*([\Sigma, \partial\Sigma])] = \beta \in \Pi,$$

with  $k+1$  boundary and  $l$  interior marked points. The boundary points are labeled according to their cyclic order. Thus, an element of  $\mathcal{M}_{k+1,l}(\beta)$  is an equivalence class under reparametrization of a triple,

$$(u : (\Sigma, \partial\Sigma) \rightarrow (X, L), \vec{z} = (z_0, z_1, \dots, z_k), \vec{w} = (w_1, \dots, w_l)), \quad z_j \in \partial\Sigma, \quad w_j \in \text{int}(\Sigma),$$

where  $\Sigma$  is a genus-0 nodal Riemann surface with one boundary component. Reparametrization acts on triples  $(u, \vec{z}, \vec{w})$  by

$$\theta.(u, (z_0, \dots, z_k), (w_1, \dots, w_l)) := (u \circ \theta, (\theta^{-1}(z_0), \dots, \theta^{-1}(z_k)), (\theta^{-1}(w_1), \dots, \theta^{-1}(w_l)))$$

for all  $\theta \in \text{Aut}(\Sigma)$ . Therefore the space  $\mathcal{M}_{k+1,l}(\beta)$  carries well defined evaluation maps

$$\begin{aligned} \text{ev}_j^\beta : \mathcal{M}_{k+1,l}(\beta) &\rightarrow L, & \text{ev}_j([u, \vec{z}, \vec{w}]) &= u(z_j), & j &= 0, \dots, k, \\ \text{evi}_j^\beta : \mathcal{M}_{k+1,l}(\beta) &\rightarrow X, & \text{evi}_j([u, \vec{z}, \vec{w}]) &= u(w_j), & j &= 1, \dots, l. \end{aligned}$$

For all  $\beta \in \Pi$ ,  $k, l \geq 0$ ,  $(k, l, \beta) \notin \{(1, 0, \beta_0), (0, 0, \beta_0)\}$ , define

$$\mathbf{q}_{k,l}^\beta : C^{\otimes k} \otimes A^*(X; Q)^{\otimes l} \longrightarrow C$$

by

$$\mathbf{q}_{k,l}^\beta(\alpha_1 \otimes \cdots \otimes \alpha_k; \gamma_1 \otimes \cdots \otimes \gamma_l) := (-1)^{\varepsilon(\alpha; \gamma)} (evb_0^\beta)_* \left( \bigwedge_{j=1}^l (evi_j^\beta)^* \gamma_j \wedge \bigwedge_{j=1}^k (evb_j^\beta)^* \alpha_j \right)$$

with

$$\varepsilon(\alpha; \gamma) := \sum_{j=1}^k j(|\alpha_j| + 1) + \sum_{j=1}^l |\gamma_j| + kn + 1.$$

The case  $\mathbf{q}_{0,0}^\beta$  is understood as  $-(evb_0^\beta)_* 1$ . Set

$$\mathbf{q}_{k,l} := \sum_{\beta \in \Pi} T^\beta \mathbf{q}_{k,l}^\beta.$$

Furthermore, for  $l \geq 0$ ,  $(l, \beta) \neq (1, \beta_0), (0, \beta_0)$ , define

$$\mathbf{q}_{-1,l}^\beta : A^*(X; Q)^{\otimes l} \longrightarrow Q$$

by

$$\begin{aligned} \mathbf{q}_{-1,l}^\beta(\gamma_1 \otimes \cdots \otimes \gamma_l) &:= (-1)^{\varepsilon(\gamma)} \int_{\mathcal{M}_{0,l}(\beta)} \bigwedge_{j=1}^l (evi_j^\beta)^* \gamma_j, \\ \mathbf{q}_{-1,l}(\gamma_1 \otimes \cdots \otimes \gamma_l) &:= \sum_{\beta \in \Pi} T^\beta \mathbf{q}_{-1,l}^\beta(\gamma_1 \otimes \cdots \otimes \gamma_l). \end{aligned}$$

Define

$$\mathbf{q}_{1,0}^{\beta_0}(\alpha) := d\alpha, \quad \mathbf{q}_{0,0}^{\beta_0} := 0, \quad \mathbf{q}_{-1,1}^{\beta_0} := 0, \quad \mathbf{q}_{-1,0}^{\beta_0} := 0.$$

For future reference, it is useful to make the following observation.

**Lemma 2.3.** *For the evaluation map associated to  $\mathcal{M}_{k+1,l}(\beta)$ ,  $\dim(\text{fiber}(evb_0^\beta)) \equiv k \pmod{2}$ .*

*Proof.* Since  $L$  is orientable,  $\mu(\beta)$  is even. Therefore,

$$\dim(\text{fiber}(evb_0^\beta)) = n - 3 + \mu(\beta) + k + 1 + 2l - n = \mu(\beta_2) + k + 2l - 2 \equiv k \pmod{2}.$$

□

Lastly, define similar operations using spheres,

$$\mathbf{q}_{\emptyset,l} : A^*(X; Q)^{\otimes l} \longrightarrow A^*(X; Q),$$

as follows. For  $\beta \in H_2(X; \mathbb{Z})$  let  $\mathcal{M}_{l+1}(\beta)$  be the moduli space of stable  $J$ -holomorphic spheres with  $l+1$  marked points indexed from 0 to  $l$  representing the class  $\beta$ , and let  $ev_j^\beta : \mathcal{M}_{l+1}(\beta) \rightarrow X$  be the evaluation maps. Assume that all the moduli spaces  $\mathcal{M}_{l+1}(\beta)$  are smooth orbifolds and  $ev_0$  is a submersion. Let  $\varpi : H_2(X; \mathbb{Z}) \rightarrow \Pi$  denote the projection. For  $l \geq 0$ ,  $(l, \beta) \neq (1, 0), (0, 0)$ , set

$$\begin{aligned} \mathbf{q}_{\emptyset,l}^\beta(\gamma_1, \dots, \gamma_l) &:= (ev_0^\beta)_* (\bigwedge_{j=1}^l (ev_j^\beta)^* \gamma_j), \\ \mathbf{q}_{\emptyset,l}(\gamma_1, \dots, \gamma_l) &:= \sum_{\beta \in H_2(X)} T^{\varpi(\beta)} \mathbf{q}_{\emptyset,l}^\beta(\gamma_1, \dots, \gamma_l), \end{aligned}$$



and define

$$\mathbf{q}_{\emptyset,1}^0 := 0, \quad \mathbf{q}_{\emptyset,0}^0 := 0.$$

*Remark 2.4.* We have assumed  $\text{evi}_0$  is a submersion. Equivalently, we could pick  $\text{evi}_j$  for  $j > 0$  to be a submersion. Then the input at the  $j$ -th place is allowed to be a current, and the resulting output is a current.

In dealing with the next result we will be using the following notation conventions.

For a fundamental list of integers  $[k] := (1, \dots, k)$ , an ordered 3-partition of  $[k]$  is a set  $\{(1:3), (2:3), (3:3)\}$  such that

$$(1:3) = (1, \dots, i_1), \quad (2:3) = (i_1 + 1, \dots, i_1 + i_2), \quad (3:3) = (i_1 + i_2 + 1, \dots, k).$$

Use  $|(i:3)|$  to denote the length of the corresponding sub-list. That is,  $|(1:3)| := i_1$ ,  $|(2:3)| := i_2$ , and  $|(3:3)| = k - i_1 - i_2$ . Denote the set of all ordered 3-partitions of  $[k]$  by  $S_3[k]$ . Similarly, denote by  $S_2[k]$  the set of ordered 2-partitions of  $[k]$ .

For a list  $\alpha = (\alpha_1, \dots, \alpha_k)$  and any (ordered) sub-list of indices  $I \leq [k]$ , write  $\alpha^I$  for the ordered sub-list of  $\alpha$  with indices in  $I$ . Write  $|\alpha^I|$  for  $\sum_{i \in I} |\alpha_i|$ . In the special case  $I = [k]$  write simply  $|\alpha| := |\alpha^I|$ .

Let  $I \sqcup J = [l]$  be a partition of  $[l]$  into two ordered sub-lists, and let  $\gamma = (\gamma_1, \dots, \gamma_l)$  be a list of differential forms. Define  $\sigma_{I \sqcup J}^\gamma$  to be the permutation that reorders  $(\gamma^I, \gamma^J)$  to  $\gamma$ . In particular,

$$\text{sgn}(\sigma_{I \sqcup J}^\gamma) \equiv \sum_{\substack{i \in I, j \in J \\ j < i}} |\gamma_i| \cdot |\gamma_j| \pmod{2}.$$

**Proposition 2.5** (The  $\mathbf{q}$ -relations). *For any fixed  $\alpha = (\alpha_1, \dots, \alpha_k)$ ,  $\gamma = (\gamma_1, \dots, \gamma_l)$ ,*

$$\begin{aligned} 0 = & \sum_{\substack{S_3[l] \\ (2:3)=\{j\}}} (-1)^{|\gamma^{(1:3)}|} \mathbf{q}_{k,l}(\alpha; \gamma^{(1:3)} \otimes d\gamma_j \otimes \gamma^{(3:3)}) + \\ & + \sum_{\substack{i \in S_3[k] \\ I \sqcup J = [l]}} (-1)^{\iota(\alpha, \gamma; i, I)} \mathbf{q}_{|(1:3)|+|(3:3)|+1, |I|}(\alpha^{(1:3)} \otimes \mathbf{q}_{|(2:3)|, |J|}(\alpha^{(2:3)}; \gamma^J) \otimes \alpha^{(3:3)}; \gamma^I), \end{aligned}$$

where

$$\begin{aligned} \iota(\alpha, \gamma; i, I) = & \sum_{j \in J} \deg_D \gamma_j \cdot \sum_{j \in (1:3)} (\deg_C \alpha_j + 1) + \sum_{j \in (1:3)} (\deg_C \alpha_j + 1) + \sum_{j \in I} \deg_D \gamma_j + \text{sgn}(\sigma_{I \sqcup J}^\gamma). \end{aligned}$$

*Remark 2.6.* Since all elements of  $R$  have even degree, we have  $|\alpha| \equiv \deg_C \alpha \pmod{2}$  and

$$\iota(\alpha, \gamma; i, I) \equiv \sum_{j \in J} |\gamma_j| \cdot \sum_{j \in (1:3)} (|\alpha_j| + 1) + \sum_{j \in (1:3)} (|\alpha_j| + 1) + \sum_{j \in I} |\gamma_j| + \text{sgn}(\sigma_{I \sqcup J}^\gamma) \pmod{2}.$$

In all computations below, we will be using these versions of  $\iota$ .

A proof is given in Section 2.3 below.

Fix a closed form  $\gamma \in \mathcal{I}_Q D$  with  $\deg_D \gamma = 2$ . Define maps on  $C$  by

$$\mathbf{m}_k^{\beta, \gamma}(\otimes_{j=1}^k \alpha_j) = \sum_l \frac{1}{l!} T^\beta \mathbf{q}_{k,l}^\beta(\otimes_{j=1}^k \alpha_j; \gamma^{\otimes l}), \quad \mathbf{m}_k^\gamma(\otimes_{j=1}^k \alpha_j) = \sum_l \frac{1}{l!} \mathbf{q}_{k,l}(\otimes_{j=1}^k \alpha_j; \gamma^{\otimes l}),$$

for all  $k \geq -1, l \geq 0$ . In particular, note that  $\mathbf{m}_{-1}^\gamma \in R$ .

**Proposition 2.7** ( $A_\infty$  relations). *The operations  $\{\mathbf{m}_k^\gamma\}_{k \geq 0}$  define an  $A_\infty$  structure on  $C$ . That is,*

$$\sum_{S_3[k]} (-1)^{\sum_{j \in (1:3)} (\deg_C \alpha_j + 1)} \mathbf{m}_{k_1}^\gamma (\alpha^{(1:3)} \otimes \mathbf{m}_{k_2}^\gamma (\alpha^{(2:3)} \otimes \alpha^{(3:3)})) = 0.$$

*Proof.* Since we have assumed  $d\gamma = 0$ , this is a special case of Proposition 2.5. □

### 2.3. Proof of the structure equations.

*Proof of Proposition 2.5.* Apply Proposition 2.2 to the case  $f = \text{ev}b_0$ ,  $M = \mathcal{M}_{k,l}(\beta)$ , and

$$\xi = \bigwedge_{j=1}^l \text{evi}_j^* \gamma_j \wedge \bigwedge_{j=1}^k \text{ev}b_j^* \alpha_j.$$

Let us see how each of the elements in Stokes' theorem looks in terms of  $\mathbf{q}$ .

*First element:*  $d(f_*\xi)$ . This is

$$d((\text{ev}b_0)_*\xi) = (-1)^{\varepsilon(\alpha;\gamma)} \mathbf{q}_{1,0}^0(\mathbf{q}_{k,l}(\alpha; \gamma)).$$

*Second element:*  $f_*(d\xi)$ . This gives

$$\begin{aligned} (\text{ev}b_0)_*(d\xi) &= (\text{ev}b_0)_* \left( d \left( \bigwedge_{j=1}^l \text{evi}_j^* \gamma_j \right) \wedge \bigwedge_{j=1}^k \text{ev}b_j^* \alpha_j \right) + \\ &\quad + (-1)^{|\gamma|} (\text{ev}b_0)_* \left( \bigwedge_{j=1}^l \text{evi}_j^* \gamma_j \wedge d \left( \bigwedge_{j=1}^k \text{ev}b_j^* \alpha_j \right) \right) \\ &= (-1)^{\varepsilon(\alpha;\gamma) + 1 + |\gamma^{(1:3)}|} \sum_i \mathbf{q}_{k,l}(\alpha; \gamma^{(1:3)} \otimes d\gamma_i \otimes \gamma^{(3:3)}) + \\ &\quad + \sum_i (-1)^{\varepsilon(\alpha;\gamma) + i + \sum_{j=1}^{i-1} |\alpha_j| + |\gamma|} \mathbf{q}_{k,l}(\alpha^{(1:3)} \otimes d\alpha_i \otimes \alpha^{(3:3)}; \gamma). \end{aligned}$$

Further,

$$\mathbf{q}_{k,l}^\beta(\alpha^{(1:3)} \otimes d\alpha_i \otimes \alpha^{(3:3)}; \gamma) = \mathbf{q}_{k,l}^\beta(\alpha^{(1:3)} \otimes \mathbf{q}_{1,0}^{\beta_0}(\alpha^{(2:3)} \otimes \alpha^{(3:3)}; \gamma).$$

*Third element:*  $(f|_{\partial M})_* \omega$ . Let  $k_i, l_i, \beta_i$ , ( $i = 1, 2$ ) be such that  $k_1 + k_2 = k + 1$ ,  $l_1 + l_2 = l$ , and  $\beta_1 + \beta_2 = \beta$ . Let  $I \sqcup J = [l]$  be a partition of the interior labels such that  $|I| = l_1$  and  $|J| = l_2$ . Write  $M_1 := \mathcal{M}_{k_1+1, l_1}(\beta_1)$  and  $M_2 := \mathcal{M}_{k_2+1, l_2}(\beta_2)$ . Let  $B \subset \partial M$  be the boundary component where a disk bubbles off at the  $i$ -th boundary point, with  $k_2$  of the boundary marked points and  $l_2$  of the interior marked points. Then there exists a canonical diffeomorphism

$$\vartheta : M_1 \times_{\text{ev}b_i^{\beta_1}} \times_{\text{ev}b_0^{\beta_2}} M_2 \xrightarrow{\sim} B.$$

A computation similar to [3, Chapter 8] shows that the diffeomorphism  $\vartheta$  changes orientation by the sign  $(-1)^\delta$  for

$$\delta := k_2(k_1 - i) + i - n. \tag{2}$$

We want to show that the contribution of  $B$  amounts to

$$\mathbf{q}_{k_1, l_1}^{\beta_1}(\alpha^{(1:3)}, \mathbf{q}_{k_2, l_2}^{\beta_2}(\alpha^{(2:3)}; \gamma^J), \alpha^{(3:3)}; \gamma^I),$$

up to sign, with  $i = |(1 : 3)| + 1$ .

Consider the pull-back diagram-

$$\begin{array}{ccc} M_1 \times_L M_2 & \xrightarrow{p_2} & M_2 \\ \downarrow p_1 & & \downarrow ev_0^{\beta_2} \\ M_1 & \xrightarrow{ev_i^{\beta_1}} & L \end{array}$$

We use the notation  $ev_j^{\beta_i}, ev_i^{\beta_j}$ , for evaluation maps on  $M_i$  with  $i = 1, 2$ . Set

$$\begin{aligned} \bar{\xi} &:= \vartheta^* \xi, \\ \xi_1 &:= \bigwedge_{j \in I} (ev_j^{\beta_1})^* \gamma_j \wedge \bigwedge_{j=1}^{i-1} (ev_j^{\beta_1})^* \alpha_j \wedge \bigwedge_{j=i+1}^{k_1} (ev_0^{\beta_1})^* \alpha_{j+k_2-1}, \\ \xi_2 &:= \bigwedge_{j \in J} (ev_j^{\beta_2})^* \gamma_j \wedge \bigwedge_{j=1}^{k_2} (ev_j^{\beta_2})^* \alpha_{j+i-1}. \end{aligned}$$

Note that

$$\bar{\xi} = (-1)^{(|\alpha^{(2:3)}| + |\gamma^J|) \cdot |\alpha^{(3:3)}| + |\gamma^J| \cdot |\alpha^{(1:3)}| + sgn(\sigma_{I \cup J}^\gamma)} p_1^* \xi_1 \wedge p_2^* \xi_2.$$

By property (4),

$$(ev_i^{\beta_1})^* (ev_0^{\beta_2})_* \xi_2 = p_{1*} p_2^* \xi_2. \quad (3)$$

Using properties (2)-(3) and equation (3),

$$\begin{aligned} (f|_B)_* \xi &= (-1)^\delta (ev_0)_* \bar{\xi} \\ &= (-1)^\delta (ev_0^{\beta_1})_* p_{1*} \bar{\xi} \\ &= (-1)^{\delta + (|\alpha^{(2:3)}| + |\gamma^J|) \cdot |\alpha^{(3:3)}| + |\gamma^J| \cdot |\alpha^{(1:3)}| + sgn(\sigma_{I \cup J}^\gamma)} (ev_0^{\beta_1})_* p_{1*} (p_1^* \xi_1 \wedge p_2^* \xi_2) \\ &= (-1)^{\delta + (|\alpha^{(2:3)}| + |\gamma^J|) \cdot |\alpha^{(3:3)}| + |\gamma^J| \cdot |\alpha^{(1:3)}| + sgn(\sigma_{I \cup J}^\gamma)} (ev_0^{\beta_1})_* (\xi_1 \wedge p_{1*} p_2^* \xi_2) \\ &= (-1)^{\delta + (|\alpha^{(2:3)}| + |\gamma^J|) \cdot |\alpha^{(3:3)}| + |\gamma^J| \cdot |\alpha^{(1:3)}| + sgn(\sigma_{I \cup J}^\gamma)} (ev_0^{\beta_1})_* \left( \xi_1 \wedge (ev_i^{\beta_1})^* (ev_0^{\beta_2})_* \xi_2 \right) \\ &= (-1)^{\delta + (|\alpha^{(2:3)}| + |\gamma^J|) \cdot |\alpha^{(3:3)}| + |\gamma^J| \cdot |\alpha^{(1:3)}| + |(ev_0^{\beta_2})_* \xi_2| \cdot |\alpha^{(3:3)}| + sgn(\sigma_{I \cup J}^\gamma)} (ev_0^{\beta_1})_* \left( \bigwedge_{j \in I} (ev_j^{\beta_1})^* \gamma_j \wedge \right. \\ &\quad \left. \wedge \bigwedge_{j=1}^{i-1} (ev_j^{\beta_1})^* \alpha_j \wedge (ev_i^{\beta_1})^* (ev_0^{\beta_2})_* \xi_2 \wedge \bigwedge_{j=i+1}^{k_1} (ev_0^{\beta_1})^* \alpha_j \right) \\ &= (-1)^* \mathbf{q}_{k_1, l_1}^{\beta_1} (\alpha^{(1:3)} \otimes \mathbf{q}_{k_2, l_2}^{\beta_2} (\alpha^{(2:3)}; \gamma^J) \otimes \alpha^{(3:3)}; \gamma^I) \end{aligned}$$

with

$$\begin{aligned} * &= \delta + (|\alpha^{(2:3)}| + |\gamma^J|) \cdot |\alpha^{(3:3)}| + |\gamma^J| \cdot |\alpha^{(1:3)}| + |(ev_0^{\beta_2})_* \xi_2| \cdot |\alpha^{(3:3)}| + \\ &\quad + \varepsilon(\alpha^{(1:3)}, (ev_0^{\beta_2})_* \xi_2, \alpha^{(3:3)}; \gamma^I) + \varepsilon(\alpha^{(2:3)}; \gamma^J) + sgn(\sigma_{I \cup J}^\gamma). \end{aligned}$$

By Lemma 2.3,

$$|(ev_0^{\beta_2})_* \xi_2| = |\alpha^{(2:3)}| + |\gamma^J| - \dim(\text{fiber}(ev_0)) \equiv |\alpha^{(2:3)}| + |\gamma^J| + k_2 \pmod{2}.$$

Again by Lemma 2.3, the dimension of the fiber of  $evb_0^\beta|_B$  is  $k \pmod{2}$ , and  $|\omega| = |\alpha| + |\gamma|$ . Therefore, the contribution of  $(f|_B)_*\xi$  to Stokes' theorem comes with the sign  $(-1)^{|\alpha|+|\gamma|+k}$ . We claim that

$$-(-1)^{|\alpha|+|\gamma|+k}(f|_B)_*\xi = (-1)^{\varepsilon(\alpha;\gamma)+\iota(\alpha,\gamma;i,I)}\mathbf{q}_{k_1,l_1}^{\beta_1}(\alpha^{(1:3)} \otimes \mathbf{q}_{k_2,l_2}^{\beta_2}(\alpha^{(2:3)}; \gamma^J) \otimes \alpha^{(3:3)}; \gamma^I).$$

Indeed,

$$\begin{aligned} & * + |\alpha| + |\gamma| + k + 1 + \varepsilon(\alpha; \gamma) \pmod{2} \equiv \\ & \equiv \varepsilon(\alpha; \gamma) + \delta + |\gamma^J| \cdot |\alpha^{(1:3)}| + k_2 \cdot |\alpha^{(3:3)}| + \\ & \quad + \varepsilon(\alpha^{(1:3)}, (evb_0^{\beta_2})_*\xi_2, \alpha^{(3:3)}; \gamma^I) + \varepsilon(\alpha^{(2:3)}; \gamma^J) + sgn(\sigma_{I \cup J}^\gamma) + |\alpha| + |\gamma| + k + 1 \\ & \equiv \sum_{j=1}^k j(|\alpha_j| + 1) + kn + 1 + k_2(k_1 - i) + i - n + |\gamma^J| \cdot |\alpha^{(1:3)}| + k_2|\alpha^{(3:3)}| + \\ & \quad + \sum_{j=1}^{i-1} j(|\alpha_j| + 1) + i \left( \sum_{j=i}^{i+k_2-1} (|\alpha_j| + 1) + |\gamma^J| + 1 \right) + \sum_{j=i+1}^{k_1} j(|\alpha_{j+k_2-1}| + 1) + k_1n + \\ & \quad + 1 + \sum_{j=1}^{k_2} j(|\alpha_{j+i-1}| + 1) + k_2n + 1 + \sum_{j=1}^k (|\alpha_j| + 1) + 1 + |\gamma^J| + |\gamma^I| + sgn(\sigma_{I \cup J}^\gamma). \end{aligned}$$

Changing variables in summations to obtain  $|\alpha_j|$  everywhere, we continue

$$\begin{aligned} & = k_2(k_1 - i) + \sum_{j=i+k_2}^k k_2 \cdot |\alpha_j| + \\ & \quad + \sum_{j=1}^{i-1} j(|\alpha_j| + 1) + \sum_{j=i}^{i+k_2-1} i(|\alpha_j| + 1) + \sum_{j=i+k_2}^k (j - k_2 + 1)(|\alpha_j| + 1) + \\ & \quad + \sum_{j=i}^{i+k_2-1} (j - i + 1)(|\alpha_j| + 1) + \sum_{j=1}^k j(|\alpha_j| + 1) + \sum_{j=1}^k (|\alpha_j| + 1) + \\ & \quad + |\gamma^J| \cdot |\alpha^{(1:3)}| + |\gamma^J|(i - 1) + |\gamma^I| + sgn(\sigma_{I \cup J}^\gamma) \\ & \equiv k_2(k_1 - i) + \sum_{j=1}^k (j + 1)(|\alpha_j| + 1) + \sum_{j=1}^{i-1} j(|\alpha_j| + 1) + \\ & \quad + \sum_{j=i}^{i+k_2-1} (j + 1)(|\alpha_j| + 1) + \sum_{j=i+k_2}^k (j + 1)|\alpha_j| + \sum_{j=i+k_2}^k (j - k_2 + 1) + \\ & \quad + |\gamma^J| \cdot \sum_{j \in (1:3)} (|\alpha_j| + 1) + |\gamma^I| + sgn(\sigma_{I \cup J}^\gamma) \\ & \equiv k_2(k_1 - i) + \sum_{j=1}^{i-1} (|\alpha_j| + 1) + \sum_{j=i+k_2}^k (j + 1) + \sum_{j=i+k_2}^k (j - k_2 + 1) + \\ & \quad + |\gamma^J| \cdot \sum_{j \in (1:3)} (|\alpha_j| + 1) + |\gamma^I| + sgn(\sigma_{I \cup J}^\gamma) \end{aligned}$$

$$\begin{aligned}
&\equiv \sum_{j=1}^{i-1} (|\alpha_j| + 1) + |\gamma^J| \cdot \sum_{j \in (1:3)} (|\alpha_j| + 1) + |\gamma^I| + \text{sgn}(\sigma_{I \sqcup J}^\gamma) \\
&= \iota(\alpha, \gamma; i, I).
\end{aligned}$$

Note that  $k_1 \geq 1$  since there is one boundary bubbling. Note also that stability of each of the bubbles implies that

$$(\beta_1, k_1, l_1) \neq (\beta_0, 1, 0), \quad (\beta_2, k_2, l_2) \neq (\beta_0, 1, 0), (\beta_0, 0, 0).$$

So, the total contribution of the summand  $(-1)^{s+t+1} (f|_{\partial M})_* \xi$  in Stokes' theorem is

$$\begin{aligned}
&(-1)^{\epsilon(\alpha; \gamma)} \sum_{\substack{\beta_1 + \beta_2 = \beta \\ k_1 + k_2 = k+1, k_1 \geq 1 \\ l_1 + l_2 = l \\ (\beta_1, k_1, l_1) \neq (\beta_0, 1, 0) \\ (\beta_2, k_2, l_2) \notin \{(\beta_0, 0, 0), (\beta_0, 1, 0)\}}} (-1)^{\iota(\alpha, \gamma; i, I)} \mathbf{q}_{k_1, l_1}^{\beta_1} (\alpha^{(1:3)} \otimes \mathbf{q}_{k_2, l_2}^{\beta_2} (\alpha^{(2:3)}; \gamma^J) \otimes \alpha^{(3:3)}; \gamma^I).
\end{aligned}$$

*Deducing the equations.* All that is left now is to plug the various expressions into Stokes' formula. Let us rewrite it first:

$$0 = d(f_* \xi) - f_*(d\xi) - (-1)^{s+t} (f|_{\partial M})_* \xi.$$

We showed that

$$\begin{aligned}
0 &= (-1)^{\epsilon(\alpha; \gamma)} \left( \mathbf{q}_{1,0}^0 \left( \mathbf{q}_{k,l}^\beta (\alpha; \gamma) \right) + (-1)^{|\gamma| + \sum_{j=1}^{i-1} |\alpha_j| + i+1} \mathbf{q}_{k,l}^\beta (\alpha^{(1:3)}, d\alpha_i, \alpha^{(3:3)}; \gamma) + \right. \\
&\quad + (-1)^{|\gamma^{(1:3)}|} \mathbf{q}_{k,l}^\beta (\alpha; \gamma^{(1:3)}, d\gamma_i, \gamma^{(3:3)}) + \\
&\quad + \sum_{\substack{\beta_1 + \beta_2 = \beta \\ k_1 + k_2 = k+1, k_1 \geq 1 \\ l_1 + l_2 = l \\ (\beta_1, k_1, l_1) \neq (0, 1, 0) \\ (\beta_2, k_2, l_2) \notin \{(0, 0, 0), (0, 1, 0)\}}} (-1)^{\iota(\alpha, \gamma; i, I)} \mathbf{q}_{k_1, l_1}^{\beta_1} (\alpha^{(1:3)} \otimes \mathbf{q}_{k_2, l_2}^{\beta_2} (\alpha^{(2:3)}; \gamma^J) \otimes \alpha^{(3:3)}; \gamma^I) \Big) \\
&= (-1)^{\epsilon(\alpha; \gamma)} \left( (-1)^{|\gamma^{(1:3)}|} \mathbf{q}_{k,l}^\beta (\alpha; \gamma^{(1:3)}, d\gamma_i, \gamma^{(3:3)}) + \right. \\
&\quad + \sum_{\substack{\beta_1 + \beta_2 = \beta \\ k_1 + k_2 = k+1, k_1 \geq 1 \\ l_1 + l_2 = l}} (-1)^{\iota(\alpha, \gamma; i, I)} \mathbf{q}_{k_1, l_1}^{\beta_1} (\alpha^{(1:3)} \otimes \mathbf{q}_{k_2, l_2}^{\beta_2} (\alpha^{(2:3)}; \gamma^J) \otimes \alpha^{(3:3)}; \gamma^I) \Big).
\end{aligned}$$

Dividing by  $(-1)^{\epsilon(\alpha, \gamma)}$  we get the desired equation. □

**2.4. The  $\mathbf{q}_{-1}$  case.** There also is a version of  $\mathbf{q}$ -relation for  $\mathbf{q}_{-1}$ .

**Proposition 2.8** (The  $\mathbf{q}_{-1}$ -relations). *For any fixed  $\gamma = (\gamma_1, \dots, \gamma_l)$ ,*

$$\begin{aligned}
0 &= \sum_{(2:3)=\{j\}} (-1)^{|\gamma^{(1:3)}|} \mathbf{q}_{-1, l} (\gamma^{(1:3)} \otimes d\gamma_j \otimes \gamma^{(3:3)}) - \\
&\quad - \frac{1}{2} \sum_{I \sqcup J = \{1, \dots, l\}} (-1)^{\sigma_{I \sqcup J}^\gamma + |\gamma^J|} \langle \mathbf{q}_{0, |I|} (\gamma^I), \mathbf{q}_{0, |J|} (\gamma^J) \rangle \pm \int_L i^* \mathbf{q}_{\emptyset, l} (\gamma).
\end{aligned}$$

*Proof.* By the classical Stokes' theorem,

$$0 = \int_{\mathcal{M}_{0,l}(\beta)} d(\wedge_{j=1}^l \text{evi}_j^* \gamma_j) - \int_{\partial \mathcal{M}_{0,l}(\beta)} \wedge_{j=1}^l \text{evi}_j^* \gamma_j.$$

We have

$$\int_{\mathcal{M}_{0,l}(\beta)} d(\wedge_{j=1}^l \text{evi}_j^* \gamma_j) = (-1)^{|\gamma^{(1:3)}| + (|\gamma|+1)-n+1} \mathbf{q}_{-1,l}^\beta(\gamma^{(1:3)} \otimes d\gamma_j \otimes \gamma^{(3:3)}).$$

The expression  $\int_{\partial \mathcal{M}_{0,l}(\beta)} \wedge_{j=1}^l \text{evi}_j^* \gamma_j$  consists of two types of contributions. One is  $\pm \int_L i^* \mathbf{q}_{0,l}^\beta(\gamma)$ , which comes from the degeneration of the boundary of the disk to a point. The other comes from boundary components arising from disk bubbling at a boundary point. Fix such a boundary component  $B$ . Let  $I \sqcup J$  be the partition of  $J$  into ordered sublists such that  $I$  indexes the interior points lying on one disk and  $J$  indexing the points lying on the other. Set  $l_1 := |I|$ ,  $l_2 := |J|$ , and let  $\beta_1$  and  $\beta_2$  be the degrees of the first and the second disk respectively. Then there is a diffeomorphism

$$\vartheta : \mathcal{M}_{1,l_1}(\beta_1) \times_{\text{ev}_0^{\beta_1}} \times_{\text{ev}_0^{\beta_2}} \mathcal{M}_{1,l_2}(\beta_2) \xrightarrow{\sim} B.$$

The change of orientation under  $\vartheta$ , given by equation (2) with  $i = k_1 = k_2 = 0$ , is  $(-1)^n$ . We claim

$$\int_B \wedge_{j=1}^l \text{evi}_j^* \gamma_j = (-1)^{\sigma_{I \cup J} + |\gamma_I| + n} \langle \mathbf{q}_{0,l_1}^{\beta_1}(\gamma^I), \mathbf{q}_{0,l_2}^{\beta_2}(\gamma^J) \rangle. \quad (4)$$

Then sum over  $\beta_1 + \beta_2 = \beta$  and divide the resulting equation by  $(-1)^{|\gamma|+n}$  to obtain

$$\begin{aligned} 0 &= \sum_{(2:3)=\{j\}} (-1)^{|\gamma^{(1:3)}|} \mathbf{q}_{-1,l}^\beta(\gamma^{(1:3)} \otimes d\gamma_j \otimes \gamma^{(3:3)}) + \\ &\quad - \frac{1}{2} \sum_{\substack{\beta_1 + \beta_2 = \beta \\ I \sqcup J = [l]}} (-1)^{\sigma_{I \cup J}} \langle \mathbf{q}_{0,|I|}^{\beta_1}(\gamma^I), \mathbf{q}_{0,|J|}^{\beta_2}(\gamma^J) \rangle \pm \int_L i^* \mathbf{q}_{0,l}^\beta(\gamma). \end{aligned}$$

The factor of  $1/2$  in the formula comes from choosing the order of the bubbles.

We now return to proving (4). Let  $\text{evi}_j^{\beta_i}$  and  $\text{ev}_0^{\beta_i}$  be the evaluation maps of  $M_i := \mathcal{M}_{1,l_i}(\beta_i)$ ,  $i = 1, 2$ . For convenience, set

$$\bar{\xi} = \vartheta^* \left( \bigwedge_{j=1}^l \text{evi}_j^* \gamma_j \right), \quad \xi_1 := \bigwedge_{j \in I} (\text{evi}_j^{\beta_1})^* \gamma_j, \quad \xi_2 := \bigwedge_{j \in J} (\text{evi}_j^{\beta_2})^* \gamma_j.$$

Again we have the pull-back diagram

$$\begin{array}{ccc} M_1 \times_L M_2 & \xrightarrow{p_1} & M_1 \\ \downarrow p_2 & & \downarrow \text{ev}_0^{\beta_1} \\ M_2 & \xrightarrow{\text{ev}_0^{\beta_2}} & L \end{array}$$

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By properties (3)-(4) and Lemma 2.3 we compute

$$\begin{aligned}
\int_{M_1 \times M_2} \bar{\xi} &= pt_*(\bar{\xi}) \\
&= (-1)^{sgn(\sigma_{I \cup J}^\gamma)} pt_* p_{2*} (p_1^* \xi_1 \wedge p_2^* \xi_2) \\
&= (-1)^{sgn(\sigma_{I \cup J}^\gamma) + |\xi_1| |\xi_2|} pt_* p_{2*} (p_2^* \xi_2 \wedge p_1^* \xi_1) \\
&= (-1)^{sgn(\sigma_{I \cup J}^\gamma) + |\xi_1| |\xi_2| + |\xi_2| |p_{2*} p_1^* \xi_1|} pt_* (p_{2*} p_1^* \xi_1 \wedge \xi_2) \\
&= (-1)^{sgn(\sigma_{I \cup J}^\gamma) + |\gamma^I| |\gamma^J| + |\gamma^J| |(evb_0^{\beta_1})_* \xi_1|} pt_* ((evb_0^{\beta_2})^* (evb_0^{\beta_1})_* \xi_1 \wedge \xi_2) \\
&= (-1)^{sgn(\sigma_{I \cup J}^\gamma) + |\gamma^I| |\gamma^J| + |\gamma^J| |\gamma^I|} pt_* (evb_0^{\beta_2})_* ((evb_0^{\beta_2})^* (evb_0^{\beta_1})_* \xi_1 \wedge \xi_2) \\
&= (-1)^{sgn(\sigma_{I \cup J}^\gamma)} pt_* ((evb_0^{\beta_1})_* \xi_1 \wedge (evb_0^{\beta_2})_* \xi_2) \\
&= (-1)^{sgn(\sigma_{I \cup J}^\gamma) + |\mathfrak{q}_{0,l_2}^{\beta_2}(\gamma^J)| + \varepsilon(\gamma^I) + \varepsilon(\gamma^J)} \langle \mathfrak{q}_{0,l_1}^{\beta_1}(\gamma^I), \mathfrak{q}_{0,l_2}^{\beta_2}(\gamma^J) \rangle \\
&= (-1)^{sgn(\sigma_{I \cup J}^\gamma) + |\gamma^J| + |\gamma^I| + 1 + |\gamma^J| + 1} \langle \mathfrak{q}_{0,l_1}^{\beta_1}(\gamma^I), \mathfrak{q}_{0,l_2}^{\beta_2}(\gamma^J) \rangle \\
&= (-1)^{sgn(\sigma_{I \cup J}^\gamma) + |\gamma^I|} \langle \mathfrak{q}_{0,l_1}^{\beta_1}(\gamma^I), \mathfrak{q}_{0,l_2}^{\beta_2}(\gamma^J) \rangle.
\end{aligned}$$

□

### 3. PROPERTIES

**3.1. Unit of the algebra.** We show that the constant form  $1 \in A^*(L; R)$  is a unit of the  $A_\infty$  algebra  $(C, \{\mathfrak{m}_k^\gamma\}_{k \geq 0})$ .

**Proposition 3.1.** Fix  $f \in A^0(L) \otimes R$ ,  $\alpha_1, \dots, \alpha_k \in C$ , and  $\gamma_1, \dots, \gamma_l \in A^*(X; Q)$ . Then

$$\mathfrak{q}_{k+1,l}^\beta(\alpha_1, \dots, \alpha_{i-1}, f, \alpha_i, \dots, \alpha_k; \otimes_{r=1}^l \gamma_r) = \begin{cases} df, & (k+1, l, \beta) = (1, 0, \beta_0), \\ f \cdot \alpha_2, & (k+1, l, \beta) = (2, 0, \beta_0), i = 1, \\ (-1)^{|\alpha_1|} f \cdot \alpha_1, & (k+1, l, \beta) = (2, 0, \beta_0), i = 2, \\ 0, & \text{otherwise.} \end{cases}$$

In particular,  $1 \in A^0(L)$  is a strong unit for the  $A_\infty$  operations  $\mathfrak{m}^\gamma$ :

$$\mathfrak{m}_{k+1}^\gamma(\alpha_1, \dots, \alpha_{i-1}, 1, \alpha_i, \dots, \alpha_k) = \begin{cases} 0, & k \geq 2 \text{ or } k = 0, \\ \alpha_2, & k = 1, i = 1, \\ (-1)^{|\alpha_1|} \alpha_1, & k = 1, i = 2. \end{cases}$$

*Proof.* The case  $(k+1, l, \beta) = (1, 0, \beta_0)$  is true by definition.

Let  $\pi : \mathcal{M}_{k+2,l}(\beta) \rightarrow \mathcal{M}_{k+1,l}(\beta)$  be the map that forgets the  $i$ -th marked boundary point and stabilizes the resulting map. Thus, the map  $\pi$  is defined only when stabilization is possible, that is, when  $(k+1, l, \beta) \neq (2, 0, \beta_0)$ . Denote by  $evb_j^{k+2}$  and  $evi_j^{k+2}$  (resp.  $evb_j^{k+1}$  and  $evi_j^{k+1}$ ) the evaluation maps for  $\mathcal{M}_{k+2,l}(\beta)$  (resp.  $\mathcal{M}_{k+1,l}(\beta)$ ). Set

$$\xi := \bigwedge_{j=1}^l (evi_j^{k+1})^* \gamma_j \wedge \bigwedge_{j=1}^k (evb_j^{k+1})^* \alpha_j.$$

Note that

$$ev_i^{k+2} = ev_j^{k+1} \circ \pi, \quad ev_j^{k+2} = \begin{cases} ev_j^{k+1} \circ \pi, & j < i, \\ ev_{j-1}^{k+1} \circ \pi, & j > i. \end{cases}$$

Thus, writing  $g := (ev_i^{k+2})^* f$ , we have

$$\begin{aligned} \pm \mathbf{q}_{k+1,l}(\alpha_1, \dots, \alpha_{i-1}, f, \alpha_i, \dots, \alpha_k; \otimes_{r=1}^l \gamma_r) &= \\ &= (ev_0^{k+2})_*(g \wedge \pi^* \xi) = (ev_0^{k+1})_*(\pi_*(g \wedge \pi^* \xi)) = (ev_0^{k+1})_*(\pi_* g \wedge \xi), \end{aligned} \quad (5)$$

whenever  $\pi$  is defined. Since  $\pi$  need not be a submersion, the push-forward  $\pi_*$  takes forms to currents in general. However, in our case, since  $\dim \mathcal{M}_{k+2,l}(\beta) > \dim \mathcal{M}_{k+1,l}(\beta)$  and  $|g| = 0$ , we have  $|\pi_* g| < 0$ . Therefore  $\pi_* g = 0$  and the right hand side of equation (5) vanishes.

Let us see what happens when  $(k+1, l, \beta) = (2, 0, \beta_0)$ . In that case,  $ev_0 = ev_1 = ev_2$ . So,

$$\begin{aligned} \mathbf{q}_{k+1,l}(f, \alpha) &= (-1)^{1+2(|\alpha|+1)+2n+1} (ev_0)_* ev_0^*(f \wedge \alpha), \\ \mathbf{q}_{k+1,l}(\alpha, f) &= (-1)^{|\alpha|+1+2 \cdot 1+2n+1} (ev_0)_* ev_0^*(\alpha \wedge f). \end{aligned}$$

Since  $\beta = \beta_0$ , the evaluation map  $ev_0$  induces an identification of the moduli space of maps with the moduli space of stable marked disks times  $L$ . Since  $k+1 = 2$  and  $l = 0$ , the space of stable disks is a point. Hence,  $ev_0$  identifies the moduli space of maps with  $L$ . Note that the identification preserves orientation. Thus,

$$\mathbf{q}_{k+1,l}(f, \alpha) = f\alpha \quad \text{and} \quad \mathbf{q}_{k+1,l}(\alpha, f) = (-1)^{|\alpha|} f\alpha.$$

□

**3.2. Cyclic structure.** Recall the definition of the pairing (1). Note that

$$\langle \xi, \eta \rangle := (-1)^{|\eta|} \int_L \xi \wedge \eta = (-1)^{|\eta|+|\eta| \cdot |\xi|} \int_L \eta \wedge \xi = (-1)^{(|\eta|+1)(|\xi|+1)+1} \langle \eta, \xi \rangle. \quad (6)$$

**Proposition 3.2.** *For any  $\alpha_1, \dots, \alpha_{k+1} \in C$  and  $\gamma_1, \dots, \gamma_l \in A^*(X; Q)$ ,*

$$\begin{aligned} \langle \mathbf{q}_{k,l}(\alpha_1, \dots, \alpha_k; \gamma_1, \dots, \gamma_l), \alpha_{k+1} \rangle &= \\ &= (-1)^{(|\alpha_{k+1}|+1) \sum_{j=1}^k (|\alpha_j|+1)} \cdot \langle \mathbf{q}_{k,l}(\alpha_{k+1}, \alpha_1, \dots, \alpha_k; \gamma_1, \dots, \gamma_l), \alpha_k \rangle. \end{aligned}$$

*In particular,  $(C, \{\mathbf{m}_k^\gamma\}_{k \geq 0})$  is a cyclic  $A_\infty$  algebra for any  $\gamma$ .*



*Proof.* Let  $pt$  be the map from  $L$  to a point. Then

$$\begin{aligned}
\langle \mathbf{q}_{k,l}(\alpha_1, \dots, \alpha_k; \gamma_1, \dots, \gamma_l), \alpha_{k+1} \rangle &= \\
&= (-1)^{|\alpha_{k+1}|} pt_* (\mathbf{q}_{k,l}(\alpha_1, \dots, \alpha_k; \gamma_1, \dots, \gamma_l) \wedge \alpha_{k+1}) \\
&= (-1)^{|\alpha_{k+1}| + \varepsilon(\alpha; \gamma)} pt_* ((evb_0)_* (\wedge_{j=1}^l evi_j^* \gamma_j \wedge \wedge_{j=1}^k evb_j^* \alpha_j) \wedge \alpha_{k+1}) \\
&= (-1)^{|\alpha_{k+1}| + \varepsilon(\alpha; \gamma) + |\alpha_{k+1}| \cdot (\sum_{j=1}^k |\alpha_j| + |\gamma| + k)} pt_* (\alpha_{k+1} \wedge (evb_0)_* (\wedge_{j=1}^l evi_j^* \gamma_j \wedge \wedge_{j=1}^k evb_j^* \alpha_j)) \\
&= (-1)^{|\alpha_{k+1}| + \varepsilon(\alpha; \gamma) + |\alpha_{k+1}| \cdot (\sum_{j=1}^k |\alpha_j| + |\gamma| + k)} pt_* (evb_0)_* (evb_0^* \alpha_{k+1} \wedge \wedge_{j=1}^l evi_j^* \gamma_j \wedge \wedge_{j=1}^k evb_j^* \alpha_j) \\
&= (-1)^{|\alpha_{k+1}| + \varepsilon(\alpha; \gamma) + |\alpha_{k+1}| \cdot (\sum_{j=1}^k |\alpha_j| + |\gamma| + k) + |\alpha_k| \cdot (|\alpha_{k+1}| + |\gamma| + \sum_{j=1}^{k-1} |\alpha_j|) + |\alpha_{k+1}| |\gamma|} \\
&\quad \cdot (pt \circ evb_0)_* (evb_k^* \alpha_k \wedge \wedge_{j=1}^l evi_j^* \gamma_j \wedge evb_0^* \alpha_{k+1} \wedge_{j=1}^{k-1} evb_j^* \alpha_j) \\
&= (-1)^{|\alpha_{k+1}| + \varepsilon(\alpha; \gamma) + |\alpha_{k+1}| \cdot (\sum_{j=1}^k |\alpha_j| + k) + |\alpha_k| \cdot (|\alpha_{k+1}| + |\gamma| + \sum_{j=1}^{k-1} |\alpha_j|)} \\
&\quad \cdot (pt \circ evb_k)_* (evb_k^* \alpha_k \wedge \wedge_{j=1}^l evi_j^* \gamma_j \wedge evb_0^* \alpha_{k+1} \wedge_{j=1}^{k-1} evb_j^* \alpha_j) \\
&= (-1)^{|\alpha_{k+1}| + \varepsilon(\alpha; \gamma) + |\alpha_{k+1}| \cdot (\sum_{j=1}^k |\alpha_j| + k) + |\alpha_k| \cdot (|\alpha_{k+1}| + |\gamma| + \sum_{j=1}^{k-1} |\alpha_j|)} \\
&\quad \cdot pt_* (\alpha_k \wedge evb_k^* (\wedge_{j=1}^l evi_j^* \gamma_j \wedge evb_0^* \alpha_{k+1} \wedge_{j=1}^{k-1} evb_j^* \alpha_j)) \\
&= (-1)^{|\alpha_{k+1}| + \varepsilon(\alpha; \gamma) + |\alpha_{k+1}| \cdot (\sum_{j=1}^k |\alpha_j| + k) + |\alpha_k| \cdot (|\alpha_{k+1}| + |\gamma| + \sum_{j=1}^{k-1} |\alpha_j|) + |\alpha_k| \cdot (|\alpha_{k+1}| + |\gamma| + \sum_{j=1}^{k-1} |\alpha_j| + k)} \\
&\quad \cdot pt_* (evb_k^* (\wedge_{j=1}^l evi_j^* \gamma_j \wedge evb_0^* \alpha_{k+1} \wedge_{j=1}^{k-1} evb_j^* \alpha_j) \wedge \alpha_k).
\end{aligned}$$

To finish the proof, we express the formula on the preceding line in terms of  $\mathbf{q}_{k,l}$  and  $\langle \cdot, \cdot \rangle$ . This contributes  $k + |\alpha_k|$  to the sign, where  $k$  accounts for cyclic relabeling of the boundary marked points and  $|\alpha_k|$  comes from the definition of the pairing. Thus, we continue

$$\begin{aligned}
&= (-1)^{|\alpha_{k+1}| + \varepsilon(\alpha; \gamma) + |\alpha_{k+1}| \cdot (\sum_{j=1}^k |\alpha_j| + k) + k \cdot |\alpha_k| + 1 \cdot (|\alpha_{k+1}| + 1) + \sum_{j=1}^{k-1} (j+1)(|\alpha_j| + 1) + |\gamma| + kn + 1 + k + |\alpha_k|} \\
&\quad \cdot \langle \mathbf{q}_{k,l}(\alpha_{k+1}, \alpha_1, \dots, \alpha_{k-1}; \gamma_1, \dots, \gamma_l), \alpha_k \rangle \\
&= (-1)^{\sum_{j=1}^{k-1} (|\alpha_j| + 1) + k(|\alpha_k| + 1) + |\alpha_{k+1}| \cdot (\sum_{j=1}^k |\alpha_j| + k) + k \cdot |\alpha_k| + 1 + k + |\alpha_k|} \\
&\quad \cdot \langle \mathbf{q}_{k,l}(\alpha_{k+1}, \alpha_1, \dots, \alpha_{k-1}; \gamma_1, \dots, \gamma_l), \alpha_k \rangle \\
&= (-1)^{\sum_{j=1}^{k-1} (|\alpha_j| + 1) + |\alpha_{k+1}| \cdot \sum_{j=1}^k (|\alpha_j| + 1) + 1 + |\alpha_k|} \\
&\quad \cdot \langle \mathbf{q}_{k,l}(\alpha_{k+1}, \alpha_1, \dots, \alpha_{k-1}; \gamma_1, \dots, \gamma_l), \alpha_k \rangle \\
&= (-1)^{(|\alpha_{k+1}| + 1) \sum_{j=1}^k (|\alpha_j| + 1)} \cdot \langle \mathbf{q}_{k,l}(\alpha_{k+1}, \alpha_1, \dots, \alpha_{k-1}; \gamma_1, \dots, \gamma_l), \alpha_k \rangle.
\end{aligned}$$

It remains to verify that  $d$  is also cyclic. Indeed,

$$\begin{aligned}
\langle d\alpha_1, \alpha_2 \rangle &= (-1)^{|\alpha_2|} \int_L d\alpha_1 \wedge \alpha_2 = \int_L ((-1)^{|\alpha_2|} d(\alpha_1 \wedge \alpha_2) + (-1)^{|\alpha_2| + |\alpha_1| + 1} \alpha_1 \wedge d\alpha_2) \\
&= (-1)^{|\alpha_2| + |\alpha_1| + 1 + |\alpha_1|(|\alpha_2| + 1)} \int_L d\alpha_2 \wedge \alpha_1 = (-1)^{|\alpha_2| + 1 + |\alpha_1|(|\alpha_2| + 1)} \langle d\alpha_2, \alpha_1 \rangle \\
&= (-1)^{(|\alpha_1| + 1)(|\alpha_2| + 1)} \langle d\alpha_2, \alpha_1 \rangle.
\end{aligned}$$

□

*Remark 3.3.* Intuitively, pairing  $\mathbf{q}_{k,l}$  with  $\alpha_{k+1}$  should be viewed as putting the constraint  $\alpha_{k+1}$  on  $z_0$ . The cyclic property then translates to a symmetry under cyclic relabeling of the boundary marked points.

### 3.3. Degree of structure maps.

**Proposition 3.4.** *For  $\gamma_1, \dots, \gamma_l \in C_2$ ,  $k \geq 0$ , the map*

$$\mathbf{q}_{k,l}(\ ; \gamma_1, \dots, \gamma_l) : C^{\otimes k} \longrightarrow C$$

*is of degree  $2 - k$  in  $C$ .*

*Proof.* It is enough to check that, for any  $\beta$ , the map

$$T^\beta \mathbf{q}_{k,l}^\beta(\ ; \gamma_1, \dots, \gamma_l) : C^{\otimes k} \longrightarrow C$$

is of degree  $2 - k$  in  $C$ . Indeed,

$$\begin{aligned} \deg_C T^\beta \mathbf{q}_{k,l}^\beta(\alpha_1, \dots, \alpha_k; \gamma_1, \dots, \gamma_l) &= \\ &= \mu(\beta) + \sum_{j=1}^k \deg_C \alpha_j + 2l - \dim_{\mathbb{R}} \text{fiber}(\text{ev}b_0) \\ &= \mu(\beta) + \sum_{j=1}^k \deg_C \alpha_j + 2l - (n - 3 + \mu(\beta) + k + 1 + 2l - n) \\ &= \sum_{j=1}^k \deg_C \alpha_j + 2 - k. \end{aligned}$$

The special case  $\mathbf{q}_{1,0}^{\beta_0} = d$  also aligns with the above formula, being of degree  $1 = 2 - 1$ . □

### 3.4. Symmetry.

**Proposition 3.5.** *Let  $k \geq -1$ . For any permutation  $\sigma \in S_l$ ,*

$$\mathbf{q}_{k,l}(\alpha_1, \dots, \alpha_k; \gamma_1, \dots, \gamma_l) = (-1)^{\text{sgn}(\sigma^\gamma)} \mathbf{q}_{k,l}(\alpha_1, \dots, \alpha_k; \gamma_{\sigma(1)}, \dots, \gamma_{\sigma(l)}),$$

*where*

$$\text{sgn}(\sigma^\gamma) := \sum_{\substack{i < j \\ \sigma(i) > \sigma(j)}} |\gamma_i| \cdot |\gamma_j| \pmod{2}.$$

*Proof.* First note that  $\varepsilon(\alpha; \gamma) = \varepsilon(\alpha; \sigma(\gamma))$ . Besides, changing the labeling of interior marked points does not affect the orientation of the moduli space. So, for  $k \geq 0$ ,

$$\begin{aligned} \mathbf{q}_{k,l}(\alpha_1, \dots, \alpha_k; \gamma_1, \dots, \gamma_l) &= (-1)^{\varepsilon(\alpha; \gamma)} (\text{ev}b_0^\beta)_* \left( \bigwedge_{j=1}^l (\text{ev}i_j^\beta)^* \gamma_j \wedge \bigwedge_{j=1}^k (\text{ev}b_j^\beta)^* \alpha_j \right) \\ &= (-1)^{\varepsilon(\alpha; \gamma) + \text{sgn}(\sigma^\gamma)} (\text{ev}b_0^\beta)_* \left( \bigwedge_{j=1}^l (\text{ev}i_{\sigma(j)}^\beta)^* \gamma_{\sigma(j)} \wedge \bigwedge_{j=1}^k (\text{ev}b_j^\beta)^* \alpha_j \right) \\ &= (-1)^{\text{sgn}(\sigma^\gamma)} \mathbf{q}_{k,l}(\alpha_1, \dots, \alpha_k; \gamma_{\sigma(1)}, \dots, \gamma_{\sigma(l)}). \end{aligned}$$

The case  $k = -1$  is similar, with  $pt$  instead of  $\text{ev}b_0^\beta$  everywhere.

□

### 3.5. Fundamental class.

**Proposition 3.6.** *For  $k \geq 0$ ,*

$$\mathfrak{q}_{k,l}^\beta(\alpha_1, \dots, \alpha_k; 1, \gamma_1, \dots, \gamma_{l-1}) = \begin{cases} -1, & (k, l, \beta) = (0, 1, \beta_0), \\ 0, & \text{otherwise.} \end{cases}$$

*Furthermore,*

$$\mathfrak{q}_{-1,l}^\beta(1, \gamma_1, \dots, \gamma_{l-1}) = 0.$$

*Proof.* Whenever defined, consider  $\pi : \mathcal{M}_{k+1,l}(\beta) \rightarrow \mathcal{M}_{k+1,l-1}(\beta)$ , the forgetful map that forgets the first interior marked point. As in the proof of Proposition 3.1, we have

$$\mathfrak{q}_{k,l}^\beta(\alpha_1, \dots, \alpha_k; 1, \gamma_1, \dots, \gamma_{l-1}) = 0$$

whenever  $\pi$  is defined. It is not defined only when forgetting the point will result in a non-stabilizable curve. This happens exactly when  $\beta = \beta_0$  and  $(k, l) \in \{(0, 1), (1, 1), (-1, 2)\}$ .

The case  $(k, l, \beta) = (1, 1, \beta_0)$  is treated as follows. Since the stable maps in  $\mathcal{M}_{2,1}(\beta_0)$  are constant, we have

$$evb_0^{\beta_0} = evb_1^{\beta_0}, \quad evi_1 = i \circ evb_0.$$

So,

$$\mathfrak{q}_{1,1}^{\beta_0}(\alpha_1; \gamma_1) = (-1)^{|\alpha_1|+1+|\gamma_1|+1 \cdot n+1} (evb_0)_* evb_0^*(i^* \gamma_1 \wedge \alpha_1) = (-1)^{|\alpha_1|+|\gamma_1|+n} i^* \gamma_1 \wedge \alpha_1 \wedge (evb_0)_* 1.$$

But  $\dim(\text{fiber}(evb_0)) = n - 3 + \mu(\beta_0) + k + 1 + 2l - n > 0$ , so  $(evb_0)_* 1 = 0$ .

The case  $(k, l, \beta) = (-1, 2, \beta_0)$  corresponds to the moduli space  $\mathcal{M}_{0,2}(\beta_0)$ . Again

$$evi_1 = evi_2 =: ev, \quad ev = i \circ evb,$$

and

$$\begin{aligned} \mathfrak{q}_{-1,2}^{\beta_0}(\gamma_1, \gamma_2) &= (-1)^{|\gamma_1|+|\gamma_2|+n+1} pt_* ev^*(\gamma_1 \wedge \gamma_2) \\ &= (-1)^{|\gamma_1|+|\gamma_2|+n+1} pt_* ev_* ev^*(\gamma_1 \wedge \gamma_2) \\ &= (-1)^{|\gamma_1|+|\gamma_2|+n+1} pt_* ((\gamma_1 \wedge \gamma_2) \wedge ev_* 1) \\ &= (-1)^{|\gamma_1|+|\gamma_2|+n+1} pt_* ((\gamma_1 \wedge \gamma_2) \wedge i_* evb_* 1). \end{aligned}$$

But  $\dim(\text{fiber}(evb)) = n - 3 + \mu(\beta_0) + 2l - n > 0$ , so  $evb_* 1 = 0$ .

The only case left is  $(0, 1, \beta_0)$ , which corresponds to the moduli space  $\mathcal{M}_{1,1}(\beta_0)$ . As in the proof of Proposition 3.1, the evaluation map  $evb_0$  identifies the moduli space of maps with  $L$ , preserving orientation. Using this identification, we see that

$$\mathfrak{q}_{0,1}^{\beta_0}(1) = -(evb_0)_* evb_0^* i^* 1 = -\text{Id}_* \text{Id}^* 1 = -1.$$

□

### 3.6. Energy zero.

**Proposition 3.7.** *For  $k \geq 0$ ,*

$$\mathfrak{q}_{k,l}^{\beta_0}(\alpha_1, \dots, \alpha_k; \gamma_1, \dots, \gamma_l) = \begin{cases} d\alpha_1, & (k, l) = (1, 0), \\ (-1)^{|\alpha_1|} \alpha_1 \wedge \alpha_2, & (k, l) = (2, 0), \\ (-1)^{|\gamma_1|+1} \gamma_1|_L, & (k, l) = (0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore,

$$\mathfrak{q}_{-1,l}^{\beta_0}(\gamma_1, \dots, \gamma_l) = 0.$$

*Proof.* The case  $\mathfrak{q}_{1,0}^{\beta_0} = d$  is true by definition. Let us consider the cases where  $\mathfrak{q}$  is defined by push-pull operations.

Since the stable maps in  $\mathcal{M}_{k,l}(\beta_0)$  are constant, we have

$$evb_0 = \dots = evb_k =: ev, \quad evi_1 = \dots = evi_l = i \circ ev.$$

Thus, for  $k \geq 0$ ,

$$\begin{aligned} \mathfrak{q}_{k,l}^{\beta_0}(\alpha_1, \dots, \alpha_k; \gamma_1, \dots, \gamma_l) &= (-1)^{\varepsilon(\alpha; \gamma)} ev_* ev^* (\wedge_{j=1}^l i^* \gamma_j \wedge \wedge_{j=1}^k \alpha_j) \\ &= (-1)^{\varepsilon(\alpha; \gamma)} (\wedge_{j=1}^l \gamma_j|_L \wedge \wedge_{j=1}^k \alpha_j) \wedge ev_* 1. \end{aligned}$$

For  $k = -1$ ,

$$\begin{aligned} \mathfrak{q}_{-1,l}^{\beta_0}(\gamma_1, \dots, \gamma_l) &= (-1)^{\varepsilon(\gamma)} pt_*(i \circ ev)_*(i \circ ev)^* (\wedge_{j=1}^l \gamma_j) \\ &= (-1)^{\varepsilon(\gamma)} pt_*((\wedge_{j=1}^l \gamma_j) \wedge i_* ev_* 1). \end{aligned}$$

In order for  $ev_* 1$  to be nonzero, we need

$$0 = \dim(\text{fiber}(ev)) = n - 3 + \mu(\beta_0) + k + 1 + 2l - n = k + 2l - 2.$$

If  $l = 1$ , then  $k = 0$ ,  $ev : \mathcal{M}_{1,1}(\beta_0) \xrightarrow{\sim} L$ , and  $\mathfrak{q}_{k,l}^{\beta_0}(\gamma_1) = (-1)^{|\gamma_1|+1} \gamma_1|_L$  by the above computation.

If  $l = 0$ , then  $k = 2$ ,  $ev : \mathcal{M}_{3,0}(\beta_0) \xrightarrow{\sim} L$ , and again by the computation above

$$\mathfrak{q}_{k,l}^{\beta_0}(\alpha_1, \alpha_2) = (-1)^{|\alpha_1|+1+2(|\alpha_2|+1)+2n+1} \alpha_1 \wedge \alpha_2.$$

□

*Remark 3.8.* Since  $\mathfrak{q}_{2,0}^{\beta_0} = \pm \wedge$ , we think of  $\mathfrak{m}_2$  as a deformation of the wedge product.

### 3.7. Divisors.

**Proposition 3.9.** *Assume  $\gamma_1|_L = 0$ ,  $|\gamma_1| = 2$ , and  $d\gamma_1 = 0$ . Then*

$$\mathfrak{q}_{k,l}^{\beta}(\otimes_{j=1}^k \alpha_j; \otimes_{j=1}^l \gamma_j) = \left( \int_{\beta} \gamma_1 \right) \cdot \mathfrak{q}_{k,l-1}^{\beta}(\otimes_{j=1}^k \alpha_j; \otimes_{j=2}^l \gamma_j). \quad (7)$$

The above also holds for the case  $k = -1$  in the obvious sense.

The proof requires the following result, which will be proved after the main proposition.

**Lemma 3.10.** *Let  $M$  be an orbifold with corners, and  $\alpha$  a degree-0 current on  $M$ . Suppose there is a current  $f$  on  $\partial M$  such that for any  $\eta \in A^{top-1}(M)$ ,*

$$\alpha(d\eta) = f(i_{\partial M}^* \eta), \quad i_{\partial M} : \partial M \longrightarrow M.$$

*Then there is a constant  $\kappa \in \mathbb{R}$  such that*

$$\alpha(\gamma) = \kappa \cdot \int_M \gamma \quad \forall \gamma \in A^{top}(M).$$

*Proof of Proposition 3.9.* Denote by  $\pi : \mathcal{M}_{k+1,l}(\beta) \rightarrow \mathcal{M}_{k+1,l-1}(\beta)$  the map that forgets the first interior marked point and shifts the labels of the others down by one. Denote by  $evb_j^l, evi_j^l$ , the evaluation maps for  $\mathcal{M}_{k+1,l}(\beta)$ , and denote by  $evb_j^{l-1}, evi_j^{l-1}$ , the evaluation maps for  $\mathcal{M}_{k+1,l-1}(\beta)$ .

For  $k \geq 0$ , set  $\xi := \wedge_{j=2}^l (evi_{j-1}^{l-1})^* \gamma_j \wedge \wedge_{j=1}^k (evb_j^{l-1})^* \alpha_j$ . Then,

$$\begin{aligned} \mathfrak{q}_{k,l}^\beta(\alpha_1, \dots, \alpha_k; \gamma_1, \dots, \gamma_l) &= (-1)^{\varepsilon(\alpha; \gamma)} (evb_0^l)_* ((evi_1^l)^* \gamma_1 \wedge \pi^* \xi) \\ &= (-1)^{\varepsilon(\alpha; \gamma) + |\xi| \cdot |\gamma_1|} (evb_0^{l-1})_* \pi_* (\pi^* \xi \wedge (evi_1^l)^* \gamma_1) \\ &= (-1)^{\varepsilon(\alpha; \gamma)} (evb_0^{l-1})_* (\xi \wedge \pi_* (evi_1^l)^* \gamma_1). \end{aligned} \quad (8)$$

Similarly, for  $k = -1$ , set  $\xi := \wedge_{j=2}^l (evi_{j-1}^{l-1})^* \gamma_j$  and compute

$$\mathfrak{q}_{-1,l}^\beta(\gamma_1, \dots, \gamma_l) = (-1)^{\varepsilon(\gamma)} pt_*(\xi \wedge \pi_* (evi_1^l)^* \gamma_1). \quad (9)$$

Since  $\pi$  is not generally a submersion, pushing forward a differential form along it results not in a differential form but rather a current. We claim that the current  $\pi_* (evi_1^l)^* \gamma_1$  acts as multiplication by  $\gamma_1(\beta)$ . To see this, decompose the codimension-1 boundary,

$$\partial \mathcal{M}_{k+1,l}(\beta) = \partial^{\text{hor}} \mathcal{M}_{k+1,l}(\beta) \coprod \partial^{\text{vert}} \mathcal{M}_{k+1,l}(\beta),$$

where  $\partial^{\text{hor}} \mathcal{M}_{k+1,l}(\beta)$  is the part of the boundary that does not require stabilization after forgetting  $w_1$ , and  $\partial^{\text{vert}} \mathcal{M}_{k+1,l}(\beta)$  is where  $w_1$  is located on a ghost bubble with one nodal boundary point and no other marked points. Elements of  $\partial^{\text{vert}} \mathcal{M}_{k+1,l}(\beta)$  are mapped by  $\pi$  to interior points of  $\mathcal{M}_{k+1,l-1}(\beta)$ , whereas  $\partial^{\text{hor}} \mathcal{M}_{k+1,l}(\beta)$  is mapped to  $\partial \mathcal{M}_{k+1,l-1}(\beta)$ . Thus, we have the following commutative diagram:

$$\begin{array}{ccccc} & & i_l^{\text{hor}} & & \\ & \searrow & \xrightarrow{\quad} & \searrow & \\ \partial^{\text{hor}} \mathcal{M}_{4,3}(\beta) & \hookrightarrow & \partial \mathcal{M}_{k+1,l}(\beta) & \xrightarrow{i_l} & \mathcal{M}_{k+1,l}(\beta) \\ \downarrow \pi_\partial & & & & \downarrow \pi \\ \partial \mathcal{M}_{k+1,l-1}(\beta) & \xrightarrow{i_{l-1}} & & & \mathcal{M}_{k+1,l-1}(\beta) \end{array}$$

For short, write  $\zeta := (evi_1^l)^* \gamma_1$ ,  $M_1 := \mathcal{M}_{k+1,l}(\beta)$ , and  $M_2 := \mathcal{M}_{k+1,l-1}(\beta)$ . By definition, for arbitrary  $\eta \in A^{top-1}(\mathcal{M}_{k+1,l-1}(\beta))$ ,

$$(\pi_* \zeta)(d\eta) = \int_{M_1} \zeta \wedge \pi^* d\eta = \int_{M_1} \zeta \wedge d(\pi^* \eta) = \int_{M_1} d(\zeta \wedge \pi^* \eta) = \int_{\partial M_1} (i_l)^* (\zeta \wedge \pi^* \eta).$$

Note that  $\zeta|_{\partial^{\text{vert}} M_1} = 0$ , because  $w_1$  is located on a ghost bubble that maps entirely to  $L$ , and  $\gamma_1|_L = 0$ . So, the computation reads

$$\begin{aligned} (\pi_* \zeta)(d\eta) &= \int_{\partial^{\text{hor}} M_1} (i_l^{\text{hor}})^* (\zeta \wedge \pi^* \eta) = \int_{\partial^{\text{hor}} M_1} (i_l^{\text{hor}})^* \zeta \wedge (i_l^{\text{hor}})^* \pi^* \eta = \\ &= \int_{\partial^{\text{hor}} M_1} (i_l^{\text{hor}})^* \zeta \wedge (i_{l-1} \circ \pi_{\partial})^* \eta = \int_{\partial^{\text{hor}} M_1} (i_l^{\text{hor}})^* \zeta \wedge \pi_{\partial}^* (i_{l-1}^* \eta) = ((\pi_{\partial})_* ((i_l^{\text{hor}})^* \zeta)) (i_{l-1}^* \eta). \end{aligned}$$

By Lemma 3.10, there is a constant  $\kappa$  such that

$$(\pi_* \zeta)(\eta) = \kappa \cdot \int_{\mathcal{M}_{k+1, l-1}(\beta)} \eta, \quad \forall \eta \in A^{\text{top}}(\mathcal{M}_{k+1, l-1}(\beta)).$$

To compute the value of  $\kappa$ , consider a point  $p = [u, \vec{z}, \vec{w}] \in \mathcal{M}_{k+1, l-1}(\beta)$  that is a regular value of  $\pi$ . In a neighborhood of such  $p$ , we can calculate  $\pi_* \zeta$  as the push-forward of a differential form. To compute its value, denote by  $v : \tilde{\Sigma} \rightarrow \Sigma$  the oriented real blowup of  $\Sigma$  at  $z_0, \dots, z_k$ . As explained in the proof of [14, Lemma 4.5], there exists a canonical orientation preserving isomorphism

$$\psi : \tilde{\Sigma} \xrightarrow{\sim} \pi^{-1}(p).$$

See Figure 3.7. Moreover,  $evi_1 \circ \psi = u \circ v$ . Since  $u_*[\Sigma] = (u \circ v)_*[\tilde{\Sigma}] \in H_2(X; \mathbb{R})$ , we have

$$\kappa = \kappa(p) = (\pi_* \zeta)_p = \int_{\pi^{-1}(p)} \zeta = \int_{\pi^{-1}(p)} evi_1^* \gamma_1 = \int_{\tilde{\Sigma}} v^* u^* \gamma_1 = \int_{\beta} \gamma_1.$$

Substituting this value in (8), we get

$$\begin{aligned} \mathfrak{q}_{k, l}^{\beta}(\alpha_1, \dots, \alpha_k; \gamma_1, \dots, \gamma_l) &= (-1)^{\varepsilon(\alpha; \gamma)} \int_{\beta} \gamma_1 \cdot (evb_0^{l-1})_* \xi \\ &= (-1)^{\varepsilon(\alpha; \gamma_2, \dots, \gamma_l)} \int_{\beta} \gamma_1 \cdot (evb_0^{l-1})_* \xi = \int_{\beta} \gamma_1 \cdot \mathfrak{q}_{k, l-1}^{\beta}(\alpha_1, \dots, \alpha_k; \gamma_2, \dots, \gamma_l). \end{aligned}$$

Similarly, substituting the value of  $\kappa$  in (9), we get

$$\mathfrak{q}_{-1, l}^{\beta}(\gamma_1, \dots, \gamma_l) = (-1)^{\varepsilon(\gamma)} \int_{\beta} \gamma_1 \cdot pt_* \xi = \int_{\beta} \gamma_1 \cdot \mathfrak{q}_{-1, l}^{\beta}(\gamma_1, \dots, \gamma_l).$$

□

We return to the proof of the auxiliary lemma.

*Proof of Lemma 3.10.* Let  $\gamma, \gamma' \in A^{\text{top}}(M)$ . Then  $\gamma, \gamma' \in A^{\text{top}}(M, \partial M)$ . Assume  $[\gamma] = [\gamma'] \in H^{\text{top}}(M, \partial M)$ . Choose  $\zeta \in A^{\text{top}-1}(M, \partial M)$  such that  $\gamma - \gamma' = d\zeta$ . Then

$$\alpha(\gamma) - \alpha(\gamma') = f(\zeta|_{\partial M}) = 0,$$

so  $\alpha(\gamma) = \alpha(\gamma')$ . This shows  $\alpha(\gamma)$  depends only on the relative cohomology class of  $\gamma$ . On the other hand, by Poincaré duality, we have an isomorphism  $H^{\text{top}}(M, \partial M) \rightarrow \mathbb{R}$  given by integration over  $M$ .

□

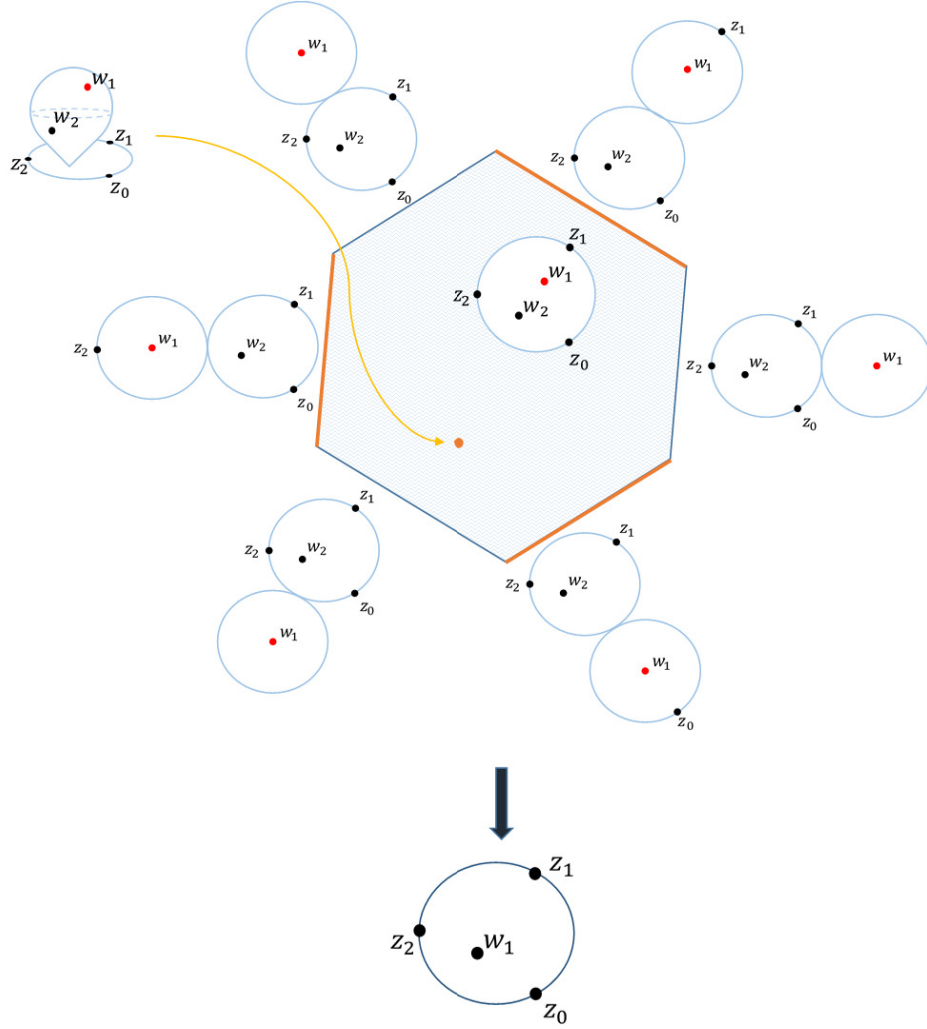


FIGURE 1. The fiber of  $\pi$  over  $[u, \vec{z}, \vec{w}]$  is the oriented real blowup of the domain at the boundary marked points. The exceptional locus of the blowup is shown in orange.

### 3.8. Top degree.

**Proposition 3.11.** *Suppose  $(k, l, \beta) \notin \{(1, 0, \beta_0), (0, 1, \beta_0), (2, 0, \beta_0)\}$ . Then  $(\mathfrak{q}_{k,l}^\beta(\alpha; \gamma))_n = 0$  for all lists  $\alpha, \gamma$ .*

*Proof.* Assume without loss of generality that  $\mathbf{q}_{k,l}^\beta(\alpha; \gamma)$  is homogeneous with respect to the grading  $|\cdot|$ . Let  $ev_j^{k+1}, ev_i_j^{k+1}$ , be the evaluation maps for  $\mathcal{M}_{k+1,l}(\beta)$ . Denote

$$\xi := \bigwedge_{j=1}^k (ev_j^{k+1})^* \alpha_j \wedge \bigwedge_{j=1}^l (ev_i_j^{k+1})^* \gamma_j,$$

that is,  $\mathbf{q}_{k,l}^\beta(\alpha; \gamma) = (-1)^{\varepsilon(\alpha; \gamma)} (ev_0^{k+1})_* \xi$ . If

$$|\mathbf{q}_{k,l}^\beta(\alpha; \gamma)| = n,$$

then

$$n = |\xi| - \dim(\text{fiber}(ev_0)) = |\xi| - (\dim \mathcal{M}_{k+1,l}(\beta) - n) = |\xi| - \dim \mathcal{M}_{k+1,l}(\beta) + n,$$

so  $|\xi| = \dim \mathcal{M}_{k+1,l}(\beta)$ .

On the other hand, if  $\pi : \mathcal{M}_{k+1,l}(\beta) \rightarrow \mathcal{M}_{k,l}(\beta)$  is the map that forgets  $z_0$ , and  $ev_j^k, ev_i_j^k$ , are the evaluation maps for  $\mathcal{M}_{k,l}(\beta)$ , then  $\xi = \pi^* \xi'$  where

$$\xi' = \bigwedge_{j=1}^k (ev_j^k)^* \alpha_j \wedge \bigwedge_{j=1}^l (ev_i_j^k)^* \gamma_j \in A^*(\mathcal{M}_{k,l}(\beta)).$$

In particular

$$\deg \xi' = \deg \xi = \dim \mathcal{M}_{k+1,l}(\beta) > \dim \mathcal{M}_{k,l}(\beta).$$

Therefore,  $\xi' = 0$  and so  $\xi = 0$ . □

**3.9. Chain map.** Write

$$T(D) := \bigoplus_{l \geq 0} D^{\otimes l}.$$

This forms a complex with the inherited differential defined as follows. For  $\eta = \bigoplus_l \eta_l$ ,  $\eta_l \in D^{\otimes l}$ , set

$$d(\eta) := \sum_i (-1)^{\sum_{l < i} |\eta_l|} \left( \bigoplus_{0 \leq l < i} \eta_l \oplus d\eta_i \oplus \bigoplus_{l > i} \eta_l \right).$$

The operators  $\mathbf{q}_{\emptyset,l}$  extend naturally to  $T(D)$  by

$$\mathbf{q}_{\emptyset}(\bigoplus_l \eta_l) := \sum_l \mathbf{q}_{\emptyset,l}(\eta_l).$$

**Proposition 3.12.** *The operator  $\mathbf{q}_{\emptyset}$  is a chain map on  $T(D)$ . That is,*

$$\mathbf{q}_{\emptyset}(d\eta) = d\mathbf{q}_{\emptyset}(\eta), \quad \forall \eta \in T(D).$$

*Proof.* Since  $\text{fiber}(ev_i_0)$  has no codimension-1 boundary, Stokes' theorem implies that  $(ev_i_0)_*$  commutes with  $d$ . □



### 3.10. Proofs of Theorems 1 and 3.

*Proof of Theorem 1.* Properties (2) and (3) are immediate from the definitions. Properties (1), (4), (5), and (6), follow respectively from Proposition 2.7, equation (6), Proposition 3.2, and Proposition 3.11. Property (8) follows directly from Proposition 3.1, while property (7) requires in addition Proposition 3.2.  $\square$

*Proof of Theorem 3.* Properties (1), (2), and (3), follow from Propositions 3.6, 3.9, and 3.7, respectively.  $\square$

## 4. PSEUDO-ISOTOPIES

**4.1. Construction.** We construct a family of  $A_\infty$  structures on  $\mathfrak{C}$ . Fix a family of  $\omega$ -tame almost complex structures  $\{J_t\}_{t \in I}$ . For each  $\beta, k, l$ , set

$$\widetilde{\mathcal{M}}_{k+1,l}(\beta) := \{(t, u) \mid u \in \mathcal{M}_{k+1,l}(\beta; J_t)\}.$$

The moduli space  $\widetilde{\mathcal{M}}_{k+1,l}(\beta)$  comes with evaluation maps

$$\begin{aligned} \widetilde{ev}b_j : \widetilde{\mathcal{M}}_{k+1,l}(\beta) &\longrightarrow I \times L, \quad j \in \{0, \dots, k\}, \\ \widetilde{ev}b_j(t, [u, \vec{z}, \vec{w}]) &:= (t, u(z_j)), \end{aligned}$$

and

$$\begin{aligned} \widetilde{evi}_j : \widetilde{\mathcal{M}}_{k+1,l}(\beta) &\longrightarrow I \times X, \quad j \in \{1, \dots, l\}, \\ \widetilde{evi}_j(t, [u, \vec{z}, \vec{w}]) &:= (t, u(w_j)). \end{aligned}$$

As with the usual moduli spaces, we assume all  $\widetilde{\mathcal{M}}_{k+1,l}(\beta)$  are smooth orbifolds with corners, and  $\widetilde{ev}b_0$  is a proper submersion.

*Example 4.1.* In the special case when  $J_t = J_0$  for all  $t \in I$ , we have

$$\widetilde{\mathcal{M}}_{k+1,l}(\beta) = I \times \mathcal{M}_{k+1,l}(\beta; J_0).$$

The evaluation maps in this case are  $\widetilde{ev}b_j = \text{Id} \times evb_j$  and  $\widetilde{evi}_j = \text{Id} \times evi_j$ . In particular, the smoothness assumptions for  $\widetilde{\mathcal{M}}_{k+1,l}(\beta)$  follow from the assumptions for  $\mathcal{M}_{k+1,l}(\beta)$ .

Even in this special case, we will see below that the moduli space  $\widetilde{\mathcal{M}}_{k+1,l}(\beta)$  allows one to prove that the  $A_\infty$  algebra  $(C, \mathfrak{m}_k^\gamma)$  for a fixed  $J$  is determined up to pseudo-isotopy by the cohomology class of  $\gamma$ .

Let

$$p : I \times L \longrightarrow I, \quad p_{\mathcal{M}} : \widetilde{\mathcal{M}}_{k+1,l}(\beta) \longrightarrow I,$$

denote the projections. Define

$$\tilde{\mathfrak{q}}_{k,l}^\beta : \mathfrak{C}^{\otimes k} \otimes A^*(I \times X; Q)^{\otimes l} \longrightarrow \mathfrak{C}, \quad k, l \geq 0,$$

by

$$\begin{aligned} \tilde{\mathfrak{q}}_{1,0}^{\beta_0=0}(\tilde{\alpha}) &= d\tilde{\alpha}, \quad \tilde{\mathfrak{q}}_{k,l}^\beta(\otimes_{j=1}^k \tilde{\alpha}_j; \otimes_{j=1}^l \tilde{\gamma}_j) := (-1)^{\varepsilon(\tilde{\alpha}, \tilde{\gamma})} (\widetilde{ev}b_0)_* (\wedge_{j=1}^k \widetilde{ev}b_j^* \tilde{\alpha}_j \wedge \wedge_{j=1}^l \widetilde{evi}_j^* \tilde{\gamma}_j), \\ \tilde{\alpha}, \tilde{\alpha}_j &\in A^*(I \times L), \quad \tilde{\gamma}_j \in A^*(I \times X). \end{aligned}$$

Define

$$\tilde{\mathfrak{q}}_{-1,l}^\beta : A^*(I \times X; Q)^{\otimes l} \longrightarrow A^*(I; Q), \quad l \geq 0,$$

by

$$\tilde{\mathfrak{q}}_{-1,l}^\beta (\otimes_{j=1}^l \tilde{\gamma}_j) := (-1)^{\varepsilon(\tilde{\gamma})} (p_{\mathcal{M}})_* \wedge_{j=1}^l \widetilde{evi_j^*} \tilde{\gamma}_j.$$

Denote by

$$\tilde{\mathfrak{q}}_{k,l} : \mathfrak{C}^{\otimes k} \otimes A^*(X; Q)^{\otimes l} \longrightarrow \mathfrak{C}, \quad \tilde{\mathfrak{q}}_{-1,l} : A^*(X; Q)^{\otimes l} \longrightarrow \mathfrak{R},$$

the sums over  $\beta$ :

$$\begin{aligned} \tilde{\mathfrak{q}}_{k,l} (\otimes_{j=1}^k \tilde{\alpha}_j; \otimes_{j=1}^l \tilde{\gamma}_j) &:= \sum_{\beta \in \Pi} T^\beta \tilde{\mathfrak{q}}_{k,l}^\beta (\otimes_{j=1}^k \tilde{\alpha}_j; \otimes_{j=1}^l \tilde{\gamma}_j), \\ \tilde{\mathfrak{q}}_{-1,l} (\otimes_{j=1}^l \tilde{\gamma}_j) &:= \sum_{\beta \in \Pi} T^\beta \tilde{\mathfrak{q}}_{-1,l}^\beta (\tilde{\gamma}^l). \end{aligned}$$

Lastly, define similar operations using spheres,

$$\tilde{\mathfrak{q}}_{\emptyset,l} : A^*(I \times X; Q)^{\otimes l} \longrightarrow A^*(I \times X; Q),$$

as follows. For  $\beta \in H_2(X; \mathbb{Z})$  let

$$\widetilde{\mathcal{M}}_{l+1}(\beta) := \{(t, u) \mid u \in \mathcal{M}_{l+1}(\beta; J_t)\}.$$

For  $j = 0, \dots, l$ , let

$$\begin{aligned} \tilde{ev}_j^\beta : \widetilde{\mathcal{M}}_{l+1}(\beta) &\rightarrow I \times X, \\ \tilde{ev}_j^\beta(t, [u, \vec{w}]) &:= (t, u(w_j)), \end{aligned}$$

be the evaluation maps. Assume that all the moduli spaces  $\widetilde{\mathcal{M}}_{l+1}(\beta)$  are smooth orbifolds and  $\tilde{ev}_0$  is a submersion. For  $l \geq 0$ ,  $(l, \beta) \neq (1, 0), (0, 0)$ , set

$$\begin{aligned} \tilde{\mathfrak{q}}_{\emptyset,l}^\beta(\tilde{\gamma}_1, \dots, \tilde{\gamma}_l) &:= (\tilde{ev}_0^\beta)_* (\wedge_{j=1}^l (\tilde{ev}_j^\beta)^* \tilde{\gamma}_j), \\ \tilde{\mathfrak{q}}_{\emptyset,l}(\tilde{\gamma}_1, \dots, \tilde{\gamma}_l) &:= \sum_{\beta \in H_2(X)} T^{\varpi(\beta)} \tilde{\mathfrak{q}}_{\emptyset,l}^\beta(\tilde{\gamma}_1, \dots, \tilde{\gamma}_l), \end{aligned}$$

and define

$$\tilde{\mathfrak{q}}_{\emptyset,1}^0 := 0, \quad \tilde{\mathfrak{q}}_{\emptyset,0}^0 := 0.$$

Define a pairing

$$\langle\langle \cdot, \cdot \rangle\rangle : \mathfrak{C} \otimes \mathfrak{C} \longrightarrow \mathfrak{R}$$

by

$$\langle\langle \tilde{\xi}, \tilde{\eta} \rangle\rangle := (-1)^{|\tilde{\eta}|} p_*(\tilde{\xi} \wedge \tilde{\eta}).$$

**Lemma 4.2.** *The operations  $\tilde{\mathfrak{q}}$  and the pairing  $\langle\langle \cdot, \cdot \rangle\rangle$  are  $\mathfrak{R}$ -linear.*

*Proof.* Let  $f \in \mathfrak{R}$ . Then

$$\begin{aligned} (\widetilde{ev}_0)_* (\widetilde{ev}_1^* (p^* f \wedge \tilde{\alpha}_1) \wedge \wedge_{j=2}^k \widetilde{ev}_j^* \tilde{\alpha}_j \wedge \wedge_{j=1}^l \widetilde{ev}_j^* \tilde{\gamma}_j) &= \\ &= (\widetilde{ev}_0)_* ((p \circ \widetilde{ev}_1)^* f \wedge \wedge_{j=1}^k \widetilde{ev}_j^* \tilde{\alpha}_j \wedge \wedge_{j=1}^l \widetilde{ev}_j^* \tilde{\gamma}_j) \\ &= (\widetilde{ev}_0)_* ((p \circ \widetilde{ev}_0)^* f \wedge \wedge_{j=1}^k \widetilde{ev}_j^* \tilde{\alpha}_j \wedge \wedge_{j=1}^l \widetilde{ev}_j^* \tilde{\gamma}_j) \\ &= (p^* f) \wedge (\widetilde{ev}_0)_* (\widetilde{ev}_1^* \tilde{\alpha}_1 \wedge \wedge_{j=2}^k \widetilde{ev}_j^* \tilde{\alpha}_j \wedge \wedge_{j=1}^l \widetilde{ev}_j^* \tilde{\gamma}_j). \end{aligned}$$

This implies linearity of  $\tilde{\mathbf{q}}$ . In addition,

$$\begin{aligned}\langle\langle p^* f \wedge \tilde{\alpha}_1, \tilde{\alpha}_2 \rangle\rangle &= p_*(p^* f \wedge \tilde{\alpha}_1 \wedge \tilde{\alpha}_2) \\ &= f \wedge p_*(\tilde{\alpha}_1 \wedge \tilde{\alpha}_2) = f \wedge \langle\langle \tilde{\alpha}_1, \tilde{\alpha}_2 \rangle\rangle.\end{aligned}$$

This gives linearity of the pairing. □

For  $t \in I$  and  $M = pt, L, X$ , denote by  $j_t : M \hookrightarrow I \times M$  the inclusion  $p \mapsto (t, p)$ . Denote by  $\mathbf{q}_{k,l}^t$  the  $\mathbf{q}$ -operators associated to the complex structure  $J_t$ .

**Lemma 4.3.** *For  $t \in I$ , we have*

$$j_t^* \tilde{\mathbf{q}}_{k,l}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k; \tilde{\gamma}_1, \dots, \tilde{\gamma}_l) = \mathbf{q}_{k,l}^t(j_t^* \tilde{\alpha}_1, \dots, j_t^* \tilde{\alpha}_k; j_t^* \tilde{\gamma}_1, \dots, j_t^* \tilde{\gamma}_l).$$

*Proof.* Consider the pull-back diagrams

$$\begin{array}{ccc} \mathcal{M}_{k+1,l}(\beta; J_t) & \xrightarrow{J_t} & \widetilde{\mathcal{M}}_{k+1,l}(\beta) \\ \downarrow \text{ev} b_i & & \downarrow \widetilde{\text{ev}} b_i \\ L & \xrightarrow{j_t} & I \times L \end{array} \quad \begin{array}{ccc} \mathcal{M}_{k+1,l}(\beta; J_t) & \xrightarrow{J_t} & \widetilde{\mathcal{M}}_{k+1,l}(\beta) \\ \downarrow \text{ev} i_i & & \downarrow \widetilde{\text{ev}} i_i \\ X & \xrightarrow{j_t} & I \times X \end{array}$$

By property (4) of integration, we have

$$\begin{aligned}j_t^*(\widetilde{\text{ev}} b_0)_*(\wedge_{i=1}^l \widetilde{\text{ev}} i_i^* \tilde{\gamma}_i \wedge \wedge_{i=1}^k \widetilde{\text{ev}} b_i^* \tilde{\alpha}_i) &= (\text{ev} b_0)_*(J_t)^*(\wedge_{i=1}^l \widetilde{\text{ev}} i_i^* \tilde{\gamma}_i \wedge \wedge_{i=1}^k \widetilde{\text{ev}} b_i^* \tilde{\alpha}_i) \\ &= (\text{ev} b_0)_*(\wedge_{i=1}^l \text{ev} i_i^* j_t^* \tilde{\gamma}_i \wedge \wedge_{i=1}^k \text{ev} b_i^* j_t^* \tilde{\alpha}_i).\end{aligned}$$

□

The next result relates the cyclic structure  $\langle\langle \cdot, \cdot \rangle\rangle$  on  $\mathfrak{C}$  with  $\langle \cdot, \cdot \rangle$  on  $C$ .

**Lemma 4.4.** *For any  $\tilde{\xi}, \tilde{\eta} \in \mathfrak{C}$ ,*

$$\int_I d\langle\langle \tilde{\xi}, \tilde{\eta} \rangle\rangle = \langle j_1^* \tilde{\xi}, j_1^* \tilde{\eta} \rangle - \langle j_0^* \tilde{\xi}, j_0^* \tilde{\eta} \rangle.$$

*Proof.* By Stokes' theorem,

$$\begin{aligned}\langle j_1^* \tilde{\xi}, j_1^* \tilde{\eta} \rangle - \langle j_0^* \tilde{\xi}, j_0^* \tilde{\eta} \rangle &= j_1^* \langle\langle \tilde{\xi}, \tilde{\eta} \rangle\rangle - j_0^* \langle\langle \tilde{\xi}, \tilde{\eta} \rangle\rangle \\ &= \int_{\partial I} \langle\langle \tilde{\xi}, \tilde{\eta} \rangle\rangle \\ &= \int_I d\langle\langle \tilde{\xi}, \tilde{\eta} \rangle\rangle.\end{aligned}$$

□

**Proposition 4.5.** *For any fixed  $\tilde{\alpha} = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_k)$ ,  $\tilde{\gamma} = (\tilde{\gamma}_1, \dots, \tilde{\gamma}_l)$ ,*

$$\begin{aligned}0 &= \sum_{\substack{S_3[l] \\ (2:3)=\{j\}}} (-1)^{|\gamma^{(1:3)}|} \tilde{\mathbf{q}}_{k,l}(\tilde{\alpha}; \tilde{\gamma}^{(1:3)} \otimes d\tilde{\gamma}_j \otimes \tilde{\gamma}^{(3:3)}) + \\ &+ \sum_{\substack{S_3[k] \\ I \sqcup J = [l]}} (-1)^{\iota(\tilde{\alpha}, \tilde{\gamma}; i, I)} \tilde{\mathbf{q}}_{|(1:3)|+(3:3)|+1, |I|}(\tilde{\alpha}^{(1:3)} \otimes \tilde{\mathbf{q}}_{|(2:3)|, |J|}(\tilde{\alpha}^{(2:3)}; \tilde{\gamma}^J) \otimes \tilde{\alpha}^{(3:3)}; \tilde{\gamma}^I).\end{aligned}$$

The proof is similar to that of Proposition 2.5.

**Proposition 4.6.** *For any fixed  $\tilde{\gamma} = (\tilde{\gamma}_1, \dots, \tilde{\gamma}_l)$ ,*

$$0 = d\tilde{\mathbf{q}}_{-1,l}(\tilde{\gamma}) + \sum_{(2:3)=\{j\}} (-1)^{|\tilde{\gamma}^{(1:3)}|} \tilde{\mathbf{q}}_{-1,l}(\tilde{\gamma}^{(1:3)} \otimes d\tilde{\gamma}_j \otimes \tilde{\gamma}^{(3:3)}) - \\ - \frac{1}{2} \sum_{I \sqcup J = \{1, \dots, l\}} (-1)^{\sigma_{I \sqcup J} + |\tilde{\gamma}_I| + n} \langle \tilde{\mathbf{q}}_{0,|I|}(\tilde{\gamma}^I), \tilde{\mathbf{q}}_{0,|J|}(\tilde{\gamma}^J) \rangle \pm p_* i^* \tilde{\mathbf{q}}_{\emptyset,l}(\tilde{\gamma}).$$

The proof is similar to that of Proposition 2.8 but uses the generalization of Stokes' theorem given in Proposition 2.2.

For each closed  $\tilde{\gamma} \in \mathcal{I}_Q \mathfrak{D}$  with  $\deg_{\mathfrak{D}} \tilde{\gamma} = 2$ , define structure maps

$$\tilde{\mathbf{m}}_k^{\tilde{\gamma}} : \mathfrak{C}^{\otimes k} \longrightarrow \mathfrak{C}$$

by

$$\tilde{\mathbf{m}}_k^{\tilde{\gamma}}(\otimes_{j=1}^k \tilde{\alpha}_j) := \sum_l \frac{1}{l!} \tilde{\mathbf{q}}_{k,l}(\otimes_{j=1}^k \tilde{\alpha}_j; \tilde{\gamma}^{\otimes l}),$$

and define

$$\tilde{\mathbf{m}}_{-1}^{\tilde{\gamma}} := \sum_l \frac{1}{l!} \tilde{\mathbf{q}}_{-1,l}(\tilde{\gamma}^{\otimes l}) \in \mathfrak{R}.$$

Denote

$$\widetilde{GW} := \sum_{l \geq 0} p_* i^* \tilde{\mathbf{q}}_{\emptyset,l}(\tilde{\gamma}^{\otimes l}).$$

**Proposition 4.7.** *The maps  $\tilde{\mathbf{m}}^{\tilde{\gamma}}$  define an  $A_{\infty}$  structure on  $\mathfrak{C}$ . That is,*

$$\sum_{\substack{k_1+k_2=k+1 \\ k_1, k_2 \geq 0 \\ 1 \leq i \leq k_1}} (-1)^{\sum_{j=1}^{i-1} (|\tilde{\alpha}_j|+1)} \tilde{\mathbf{m}}_{k_1}^{\tilde{\gamma}}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_{i-1}, \tilde{\mathbf{m}}_{k_2}^{\tilde{\gamma}}(\tilde{\alpha}_i, \dots, \tilde{\alpha}_{i+k_2-1}), \tilde{\alpha}_{i+k_2}, \dots, \tilde{\alpha}_k) = 0$$

for all  $\tilde{\alpha}_j \in A^*(I \times L)$ .

*Proof.* Since  $d\tilde{\gamma} = 0$ , this is a special case of Proposition 4.5. □

**4.2. Properties.** The properties formulated for the  $\mathbf{q}$ -operators can be equally well formulated for the  $\tilde{\mathbf{q}}$ -operators.

**Proposition 4.8** (Unit of the algebra). *Fix  $f \in A^0(I \times L) \otimes R$ ,  $\tilde{\alpha}_1, \dots, \tilde{\alpha}_k \in \mathfrak{C}$ , and  $\tilde{\gamma}_1, \dots, \tilde{\gamma}_l \in A^*(I \times X; Q)$ . Then*

$$\tilde{\mathbf{q}}_{k+1,l}^{\beta}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_{i-1}, f, \tilde{\alpha}_i, \dots, \tilde{\alpha}_k; \otimes_{r=1}^l \tilde{\gamma}_r) = \begin{cases} df, & (k+1, l, \beta) = (1, 0, \beta_0), \\ f \cdot \tilde{\alpha}_2, & (k+1, l, \beta) = (2, 0, \beta_0), i = 1, \\ (-1)^{|\tilde{\alpha}_1|} f \cdot \tilde{\alpha}_1, & (k+1, l, \beta) = (2, 0, \beta_0), i = 2, \\ 0, & \text{otherwise.} \end{cases}$$

In particular,  $1 \in A^0(I \times L)$  is a strong unit for the  $A_{\infty}$  operations  $\tilde{\mathbf{m}}^{\tilde{\gamma}}$ :

$$\tilde{\mathbf{m}}_{k+1}^{\tilde{\gamma}}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_{i-1}, 1, \tilde{\alpha}_i, \dots, \tilde{\alpha}_k) = \begin{cases} 0, & k+1 \geq 3 \text{ or } k+1 = 1, \\ \tilde{\alpha}_2, & k+1 = 2, i = 1, \\ (-1)^{|\tilde{\alpha}_1|} \tilde{\alpha}_1, & k+1 = 2, i = 2. \end{cases}$$

The proof is analogous to that of Proposition 3.1.

**Proposition 4.9** (Cyclic structure). *The  $\tilde{\mathfrak{q}}$  are cyclic with respect to the inner product  $\langle\langle \cdot, \cdot \rangle\rangle$ . That is,*

$$\begin{aligned} \langle\langle \tilde{\mathfrak{q}}_{k,l}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k; \tilde{\gamma}_1, \dots, \tilde{\gamma}_l), \tilde{\alpha}_{k+1} \rangle\rangle &= \\ &= (-1)^{(|\tilde{\alpha}_{k+1}|+1) \sum_{j=1}^k (|\tilde{\alpha}_j|+1)} \cdot \langle\langle \tilde{\mathfrak{q}}_{k,l}(\tilde{\alpha}_{k+1}, \tilde{\alpha}_1, \dots, \tilde{\alpha}_{k-1}; \tilde{\gamma}_1, \dots, \tilde{\gamma}_l), \tilde{\alpha}_k \rangle\rangle + \delta_{1,k} \cdot d\langle\langle \tilde{\alpha}_1, \tilde{\alpha}_2 \rangle\rangle. \end{aligned}$$

*Proof.* For  $(k, l, \beta) \neq (1, 0, \beta_0)$ , the proof is the same as in Proposition 3.2. In the remaining case,

$$\begin{aligned} \langle\langle d\tilde{\alpha}_1, \tilde{\alpha}_2 \rangle\rangle &= (-1)^{|\tilde{\alpha}_2|} p_*(d\tilde{\alpha}_1 \wedge \tilde{\alpha}_2) \\ &= p_*((-1)^{|\tilde{\alpha}_2|} d(\tilde{\alpha}_1 \wedge \tilde{\alpha}_2) - (-1)^{|\tilde{\alpha}_1|+|\tilde{\alpha}_2|} \tilde{\alpha}_1 \wedge d\tilde{\alpha}_2) \\ &= (-1)^{|\tilde{\alpha}_2|} d(p_*(\tilde{\alpha}_1 \wedge \tilde{\alpha}_2)) + (-1)^{|\tilde{\alpha}_1|+|\tilde{\alpha}_2|+1+|\tilde{\alpha}_1|(|\tilde{\alpha}_2|+1)} p_*(d\tilde{\alpha}_2 \wedge \tilde{\alpha}_1) \\ &= d\langle\langle \tilde{\alpha}_1, \tilde{\alpha}_2 \rangle\rangle + (-1)^{(|\tilde{\alpha}_1|+1)(|\tilde{\alpha}_2|+1)} \langle\langle d\tilde{\alpha}_2, \tilde{\alpha}_1 \rangle\rangle. \end{aligned}$$

□

**Proposition 4.10** (Degree of structure maps). *For  $\tilde{\gamma}_1, \dots, \tilde{\gamma}_l \in \mathfrak{C}_2$ ,  $k \geq 0$ , the map*

$$\tilde{\mathfrak{q}}_{k,l}(\cdot; \tilde{\gamma}_1, \dots, \tilde{\gamma}_l) : \mathfrak{C}^{\otimes k} \longrightarrow \mathfrak{C}$$

*is of degree  $2 - k$  in  $\mathfrak{C}$ .*

The proof is similar to that of Proposition 3.4.

**Proposition 4.11** (Energy zero). *For  $k \geq 0$ ,*

$$\tilde{\mathfrak{q}}_{k,l}^{\beta_0}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k; \tilde{\gamma}_1, \dots, \tilde{\gamma}_l) = \begin{cases} d\tilde{\alpha}_1, & (k, l) = (1, 0), \\ (-1)^{|\tilde{\alpha}_1|} \tilde{\alpha}_1 \wedge \tilde{\alpha}_2, & (k, l) = (2, 0), \\ (-1)^{|\tilde{\gamma}_1|+1} \tilde{\gamma}_1|_L, & (k, l) = (0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

*Furthermore,*

$$\tilde{\mathfrak{q}}_{-1,l}^{\beta_0}(\tilde{\gamma}_1, \dots, \tilde{\gamma}_l) = 0.$$

*Proof.* Note that  $\dim(\text{fiber}(\text{evi}_j)) = \dim(\text{fiber}(\widetilde{\text{evi}}_j))$  and  $\dim(\text{fiber}(\text{evb}_j)) = \dim(\text{fiber}(\widetilde{\text{evb}}_j))$  for any  $j$ . Therefore the proof of Proposition 3.7 is valid verbatim in our case, with  $\mathfrak{q}$  replaced by  $\tilde{\mathfrak{q}}$  everywhere.

□

**Proposition 4.12** (Top degree). *Assume  $(k, l, \beta) \notin \{(1, 0, \beta_0), (0, 1, \beta_0), (2, 0, \beta_0)\}$ . Then  $(\tilde{\mathfrak{q}}_{k,l}^\beta(\tilde{\alpha}; \tilde{\gamma}))_{n+1} = 0$  for all lists  $\tilde{\alpha}, \tilde{\gamma}$ .*

The proof is similar to that of Proposition 3.11. There are also analogs of Propositions 3.5, 3.6, and 3.9 for  $\tilde{\mathfrak{q}}$ . The proofs are again similar.

**Lemma 4.13.** *For all lists  $\tilde{\gamma}$ , we have  $\langle\langle \tilde{\mathfrak{q}}_{0,l}(\tilde{\gamma}), 1 \rangle\rangle = 0$ .*

*Proof.* By Proposition 4.12 it remains to check that  $(\langle\langle \tilde{\mathbf{q}}_{0,l}(\tilde{\gamma}), 1 \rangle\rangle)_0 = 0$ . To see this, we evaluate at an arbitrary point  $t \in I$ . Consider the pull-back diagram

$$\begin{array}{ccc} L & \xrightarrow{j_t} & I \times L \\ \downarrow & & \downarrow p \\ pt & \xrightarrow{j_t} & I \end{array}$$

Then by property (4) of integration, Lemma 4.3 and Proposition 3.11, we have

$$(\langle\langle \tilde{\mathbf{q}}_{0,l}(\tilde{\gamma}), 1 \rangle\rangle)_0(t) = j_t^*(p_* \tilde{\mathbf{q}}_{0,l}(\tilde{\gamma}))_0 = j_t^* p_*(\tilde{\mathbf{q}}_{0,l}(\tilde{\gamma}))_n = \int_L j_t^*(\tilde{\mathbf{q}}_{0,l}(\tilde{\gamma}))_n = \int_L \mathbf{q}_{0,l}^t(j_t^* \tilde{\gamma}) = 0.$$

□

*Proof of Theorem 2.* Choose  $\eta \in D$  with  $\deg_D \eta = 1$  such that  $\gamma - \gamma' = d\eta$ . Take

$$\tilde{\gamma} := \gamma + t(\gamma - \gamma') + dt \wedge \eta \in \mathfrak{D}.$$

Then  $\deg_{\mathfrak{D}} \tilde{\gamma} = 2$  and

$$\begin{aligned} d\tilde{\gamma} &= dt \wedge (\gamma - \gamma') - dt \wedge d\eta = 0, \\ j_0^* \tilde{\gamma} &= \gamma, \quad j_1^* \tilde{\gamma} = \gamma'. \end{aligned}$$

By Lemmas 4.2, 4.3, 4.13, and Propositions 4.7, 4.8, 4.9, 4.10,  $(\mathfrak{C}, \tilde{\mathbf{m}}^{\tilde{\gamma}})$  is a cyclic unital pseudo-isotopy from  $(C, \mathbf{m}^{\gamma})$  to  $(C, \mathbf{m}^{\gamma'})$ .

□

**4.3. Uniform formulation of structure equations.** Using the cyclic structure  $\langle\langle \cdot, \cdot \rangle\rangle$ , the  $A_{\infty}$  relations can be rephrased so the case  $k = -1$  fits more uniformly.

**Proposition 4.14** ( $A_{\infty}$  relations on an isotopy). *For  $k \geq 0$ ,*

$$\begin{aligned} d\langle\langle \tilde{\mathbf{m}}_k^{\tilde{\gamma}}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k), \tilde{\alpha}_{k+1} \rangle\rangle &= \\ &= \sum_{\substack{k_1+k_2=k+1 \\ k_1 \geq 1, k_2 \geq 0 \\ 1 \leq i \leq k_1}} (-1)^{\nu(\tilde{\alpha}; k_1, k_2, i)} \langle\langle \tilde{\mathbf{m}}_{k_1}^{\tilde{\gamma}}(\tilde{\alpha}_{i+k_2}, \dots, \tilde{\alpha}_{k+1}, \tilde{\alpha}_1, \dots, \tilde{\alpha}_{i-1}), \tilde{\mathbf{m}}_{k_2}^{\tilde{\gamma}}(\tilde{\alpha}_i, \dots, \tilde{\alpha}_{k_2+i}) \rangle\rangle \end{aligned}$$

with

$$\nu(\tilde{\alpha}; k_1, k_2, i) := \sum_{j=1}^{i-1} (|\tilde{\alpha}_j| + 1) + \sum_{j=i+k_2}^{k+1} (|\tilde{\alpha}_j| + 1) \left( \sum_{\substack{m \neq j \\ 1 \leq m \leq k+1}} (|\tilde{\alpha}_m| + 1) + 1 \right) + 1$$

For  $k = -1$ ,

$$d\tilde{\mathbf{m}}_{-1}^{\tilde{\gamma}} = (-1)^n \frac{1}{2} \langle\langle \tilde{\mathbf{m}}_0^{\tilde{\gamma}}, \tilde{\mathbf{m}}_0^{\tilde{\gamma}} \rangle\rangle \pm \widetilde{GW}.$$

*Proof.* For  $k \geq 0$ , we use Propositions 4.9 and 4.7 to obtain

$$\begin{aligned}
d\langle\langle \tilde{\mathbf{m}}_k^{\tilde{\gamma}}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k), \tilde{\alpha}_{k+1} \rangle\rangle &= \\
&= \langle\langle d\tilde{\mathbf{m}}_k^{\tilde{\gamma}}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k), \tilde{\alpha}_{k+1} \rangle\rangle - (-1)^{(|\tilde{\alpha}_{k+1}|+1)(|\tilde{\mathbf{m}}_k^{\tilde{\gamma}}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k)|+1)} \langle\langle d\tilde{\alpha}_{k+1}, \tilde{\mathbf{m}}_k^{\tilde{\gamma}}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k) \rangle\rangle \\
&= - \sum_{\substack{k_1+k_2=k+1 \\ (k_1, \beta) \neq (1, \beta_0)}} (-1)^{\sum_{j=1}^{i-1} (|\tilde{\alpha}_j|+1)} \langle\langle \tilde{\mathbf{m}}_{k_1}^{\tilde{\gamma}, \beta}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_{i-1}, \tilde{\mathbf{m}}_{k_2}^{\tilde{\gamma}}(\tilde{\alpha}_i, \dots, \tilde{\alpha}_{i+k_2-1}), \tilde{\alpha}_{i+k_2}, \dots, \tilde{\alpha}_k), \tilde{\alpha}_{k+1} \rangle\rangle + \\
&\quad + (-1)^{(|\tilde{\alpha}_{k+1}|+1)(|\tilde{\mathbf{m}}_k^{\tilde{\gamma}}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k)|+1)+1} \langle\langle d\tilde{\alpha}_{k+1}, \tilde{\mathbf{m}}_k^{\tilde{\gamma}}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k) \rangle\rangle \\
&= \sum_{\substack{k_1+k_2=k+1 \\ (k_1, \beta) \neq (1, \beta_0)}} (-1)^{\sum_{j=1}^{i-1} (|\tilde{\alpha}_j|+1)+\nu'} \langle\langle \tilde{\mathbf{m}}_{k_1}^{\tilde{\gamma}, \beta}(\tilde{\alpha}_{i+k_2}, \dots, \tilde{\alpha}_k, \tilde{\alpha}_{k+1}\tilde{\alpha}_1, \dots, \tilde{\alpha}_{i-1}), \tilde{\mathbf{m}}_{k_2}^{\tilde{\gamma}}(\tilde{\alpha}_i, \dots, \tilde{\alpha}_{i+k_2-1}) \rangle\rangle + \\
&\quad + (-1)^{(|\tilde{\alpha}_{k+1}|+1)(\sum_{j=1}^k (|\tilde{\alpha}_j|+1)+1)+1} \langle\langle d\tilde{\alpha}_{k+1}, \tilde{\mathbf{m}}_k^{\tilde{\gamma}}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k) \rangle\rangle,
\end{aligned}$$

with the sign  $\nu'$  as follows:

$$\begin{aligned}
\nu' &= 1 + (|\tilde{\alpha}_{k+1}| + 1) \left( \sum_{\substack{1 \leq j \leq k \\ j \notin \{i, \dots, i+k_2-1\}}} (|\tilde{\alpha}_j| + 1) + (|\tilde{\mathbf{m}}^{\tilde{\gamma}}(\tilde{\alpha}_i, \dots, \tilde{\alpha}_{i+k_2-1})| + 1) \right) + \\
&\quad + \sum_{j=i+k_2}^k (|\tilde{\alpha}_j| + 1) \left( \sum_{\substack{m \neq j \\ 1 \leq m \leq k+1 \\ m \notin \{i, \dots, k_2+i\}}} (|\tilde{\alpha}_m| + 1) + (|\tilde{\mathbf{m}}^{\tilde{\gamma}}(\tilde{\alpha}_i, \dots, \tilde{\alpha}_{i+k_2-1})| + 1) \right) \\
&\equiv \sum_{j=i+k_2}^{k+1} (|\tilde{\alpha}_j| + 1) \left( \sum_{\substack{m \neq j \\ 1 \leq m \leq k+1}} (|\tilde{\alpha}_m| + 1) + 1 \right) \pmod{2}.
\end{aligned}$$

For  $k = -1$ , note that  $\deg_{\mathfrak{D}} \tilde{\gamma} = 2$ , so

$$|\tilde{\gamma}^I| \equiv \text{sgn}(\sigma_{I \cup J}^{\tilde{\gamma}}) \equiv 0 \pmod{2}.$$

This implies that Proposition 4.6 reads

$$0 = d\tilde{\mathbf{m}}_{-1}^{\tilde{\gamma}} - (-1)^n \frac{1}{2} \langle\langle \tilde{\mathbf{m}}_0^{\tilde{\gamma}}, \tilde{\mathbf{m}}_0^{\tilde{\gamma}} \rangle\rangle \pm \widetilde{GW}.$$

□

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